UNIT 13

Exponentials

Introduction

People sometimes talk about things growing exponentially – the population of the world or the number of people with a particular disease, for example. What exactly does this mean?

The idea of exponential growth can be understood by thinking about chain letters (or, more commonly nowadays, chain emails or chain text messages). A typical chain letter contains a message that the recipient is asked to send on to, say, five other people. For example, the message in one spoof chain letter asked recipients to 'help a dying child get into the Guinness Book of World Records' by sending their business cards to a certain address, and recipients were asked to forward this message to several other people.

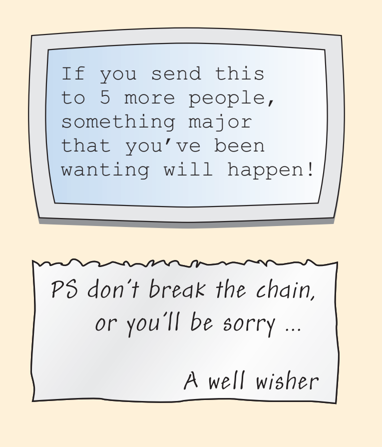


Figure 1 Typical examples of the carrot and stick approaches in chain letters, emails or text messages

To encourage recipients of a chain letter to continue the chain, they may be promised good luck if they cooperate (the 'carrot approach') or threatened with bad luck if they don't (the 'stick approach').

If each letter in a chain is sent to five recipients, each of whom sends a letter to five new people, who in turn each send it on to five more people, then the growth in the number of letters is remarkable:

first round: 5 letters;

second round: $5 \times 5 = 5^2 = 25$ letters;

third round: $5 \times 25 = 5^3 = 125$ letters;

fourth round: $5 \times 125 = 5^4 = 625$ letters;

and so on.

If all the recipients respond as asked, and each person enters the chain only once, then the twentieth round consists of 5^{20} letters, which is nearly 100 trillion – about 10 000 times the number of people in the world. Even if only a certain proportion of recipients respond, then the number of letters grows extremely rapidly. For example, if three out of the five recipients respond on each occasion, then the twentieth round consists of 3^{20} letters, which is more than three billion – about half of the number of people in the world.

This type of growth, where the numbers increase by the same factor at each stage, is called *exponential growth*. Exponential growth can be extremely rapid, but 'exponential growth' does not *mean* 'rapid growth' – it is a common misconception that the two mean the same. Exponential growth can also be slow, if the factor by which the numbers increase is small. And some types of growth are rapid but not exponential. So be aware that when you see or hear the term 'exponential growth' used in an everyday context, its user may not intend the clearly-defined mathematical meaning given in this unit.

You will learn about exponential growth in this unit, and also about *exponential decay*, which is closely related to exponential growth but involves decreasing quantities rather than increasing ones.

In Sections 1 and 2 you will learn exactly what is meant by exponential growth and decay, and you will see a range of examples of this type of change. You will also see how you can use models based on this type of change to make predictions. In Section 3 you will explore the graphs of the equations that describe exponential growth and decay, and you will be introduced to the important mathematical constant e, which has a value of

roughly 2.718 and a special place in mathematics. Finally, in Sections 4 and 5 you will meet the idea of *logarithms*. You will see that these are closely related to the ideas covered in the earlier parts of the unit, and learn about some of their uses.

The calculator section of the MU123 Guide is needed for two of the activities in this unit. If you do not have the MU123 Guide to hand when you reach these activities, then you can omit them and return to them later. However, you will need to work through these activities before you study Subsections 5.2 and 5.3.

Activities 21 and 25, on pages 153 and 156, respectively, are in the MU123 Guide.

1 Exponential growth and decay

In this section you will learn about exponential growth and decay by considering a number of examples.

1.1 Doubling and halving

The activity below asks you to think about growth in a particular situation.

Activity 1 *Guessing the height of a pile of paper*

Imagine tearing a long strip of paper in half and placing one half on top of the other, then tearing the two pieces in half again and stacking them to make a pile of four pieces, then repeating the process to make a stack of eight pieces, and so on. Suppose that you continue until you have carried out the process 50 times in all. What do you think the height of your paper pile might be, roughly?

Here is another puzzle that reveals something about the surprisingly powerful effect of doubling.

Activity 2 *Thinking about doubling*

Imagine a pond containing water lilies (Figure 2). Suppose that the area of the pond covered by the lily pads doubles every day. It takes 30 days for the lily pads to completely cover the pond. After how many days did the lily pads cover exactly half of the pond?

Figure 2 Lily pads in a pond

Another striking illustration of the effect of repeated doubling comes from a legend. It is said that the ruler of an ancient land was so delighted with the newly invented game of chess that he decided to reward its inventor. Rashly, he asked the inventor to name his own reward. The inventor cunningly asked the ruler for the following: for the first square of the chessboard (Figure 3, overleaf), he should receive one grain of rice, then two grains for the second square, four grains for the third square, and so on, doubling the number of grains each time until all 64 squares were accounted for.

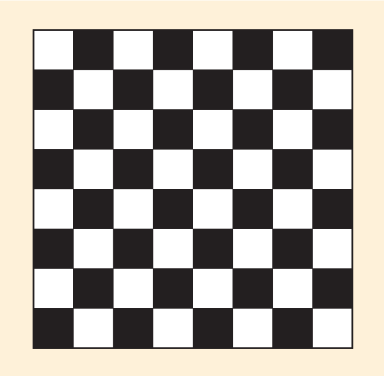


Figure 3 A chessboard, with 64 squares

The ruler foolishly agreed to this arrangement. However, his state treasurer subsequently discovered that it would be impossible to give the inventor the reward.

You can work out the number of grains of rice on the successive squares of the chessboard as follows. Let's number the squares from 0 to 63, so that square 0 contains the first grain of rice, square 1 contains the number of grains after 1 doubling, square 2 contains the number of grains after 2 doublings, and so on. Then

the number of grains on square 0 is 1;
the number of grains on square 1 is 1×2;
the number of grains on square 2 is $1 \times 2 \times 2 = 2^2$;
the number of grains on square 3 is $1 \times 2 \times 2 \times 2 = 2^3$;

and so on. In general,

the number of grains on square n is $1 \times 2^n = 2^n$.

The total number of grains of rice comes to

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{63}.$$

This is a lot of rice! In fact, it is much more than the amount that would be produced in one harvest, at modern yields, if all the Earth's arable land could be devoted to growing rice.

You have now seen several illustrations of the effect of repeated doubling. Repeated doubling always produces the type of growth called exponential growth.

To understand exactly what 'exponential growth' means, think back to the first illustration, the pile of paper. The height in metres of the initial pile was 0.0001, and each successive height was calculated by multiplying the previous height by the same number, namely 2.

Exponential growth is growth that arises from repeated multiplication by the same number. The number that you start with is called the **starting number**, and the number that you multiply by is called the **scale factor** (or *multiplication factor*). For exponential growth, the scale factor must be greater than 1, so that when you multiply you obtain an increase rather than a decrease.

You have already met the idea of a scale factor in various different contexts earlier in the module.

In the example of the paper pile, the starting number is 0.0001, since this is the height in metres of the initial pile of one strip of paper, and the scale factor is 2, since the height of the pile is multiplied by 2 at each step. As you saw in the solution to Activity 1, the successive heights in metres of the pile of paper are as follows:

the initial height is the starting number, 0.0001;
after 1 step, the height is 0.0001×2;
after 2 steps, the height is $0.0001 \times 2 \times 2 = 0.0001 \times 2^2$;
after 3 steps, the height is $0.0001 \times 2 \times 2 \times 2 = 0.0001 \times 2^3$;

and so on. In general,

after n steps, the height in metres is 0.0001×2^n.

A formula for exponential growth with any starting value and any scale factor can be worked out in the same way. If the starting value is a and the scale factor is b, then:

the initial value is the starting number, a;
after 1 step, the value is $a \times b = ab$;
after 2 steps, the value is $a \times b \times b = ab^2$;
after 3 steps, the value is $a \times b \times b \times b = ab^3$;

and so on. In general,

after n steps, the value is ab^n.

These ideas about exponential growth are summarised in the box below.

Exponential growth

Suppose that a positive quantity changes in steps, where its value at each step is obtained from its value at the previous step by multiplying by the same constant, which is greater than 1. Then the quantity is said to **grow exponentially**. The constant is called the scale factor.

If the starting number is a and the scale factor is b, then the value after n steps is ab^n.

The word *exponential* arises from the fact that the number of steps, n, is in the *exponent* in the formula ab^n.

Remember that *exponent* is another name for *power* or *index*.

Figure 4 shows the height in metres of the paper pile at the beginning of the process and after each of the first ten steps. You can see that the growth starts slowly, but keeps increasing and soon becomes very rapid indeed. This is typical of exponential growth.

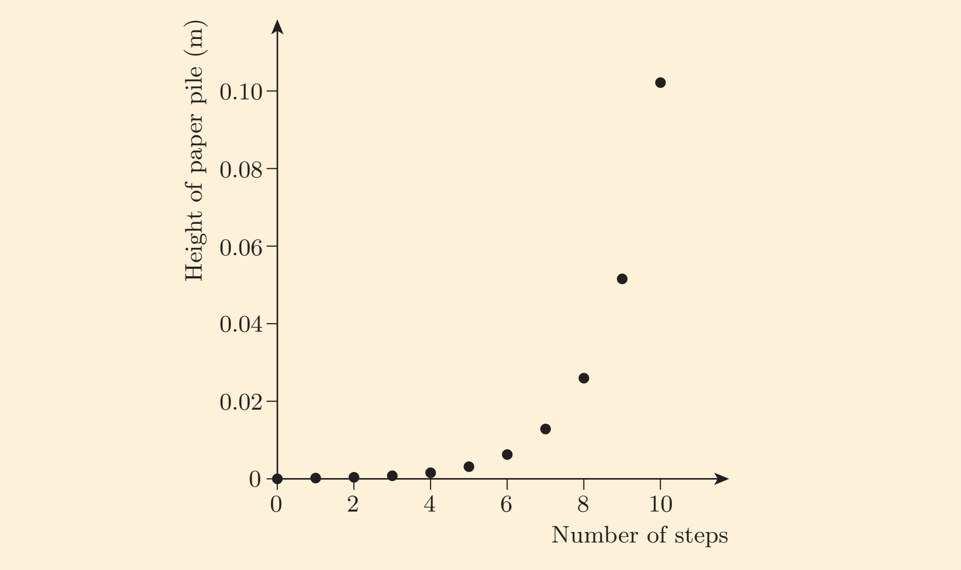
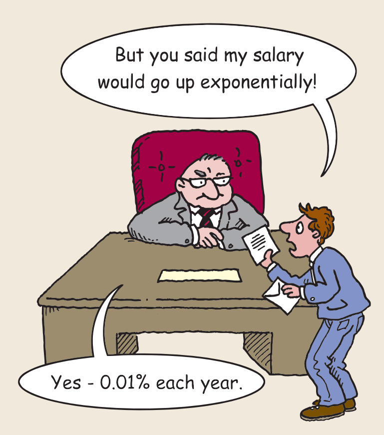


Figure 4 The height of the paper pile after each of the first ten steps



However, as you saw in the introduction to the unit, exponential growth does not necessarily mean very fast growth, contrary to how the term is often used in the media. If the scale factor were, say, 1.000 001, then the quantity would grow only very slowly, but the growth would still be exponential growth.

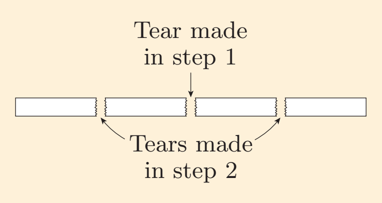


Figure 5 Tears in the paper strip at right angles to the long edges

Now let's think about the paper-tearing process from a different point of view. Suppose that you always tear the paper strips exactly in half, and always at right angles to the long edges of the initial strip of paper, as illustrated in Figure 5. Consider the length of the pieces of paper after each step. Suppose that you start with a strip of paper 0.7 metres long. Then the lengths, in metres, of the strips after each step are as follows:

the initial length is 0.7;

after 1 step, the length is $0.7 \times \frac{1}{2}$;

after 2 steps, the length is $0.7 \times \frac{1}{2} \times \frac{1}{2} = 0.7 \times \left(\frac{1}{2}\right)^2$;

after 3 steps, the length is $0.7 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 0.7 \times \left(\frac{1}{2}\right)^3$;

and so on. In general,

after n steps, the length of the pieces of paper in metres is $0.7 \times \left(\frac{1}{2}\right)^n$.

This formula is of the same form as the formula for exponential growth. It arises from a starting number, 0.7, repeatedly multiplied by a scale factor, $\frac{1}{2}$. The only difference is that here the scale factor is less than 1 (but still positive), so at each step the size *decreases* rather than increases. This is an example of *exponential decay*.

Exponential decay

Suppose that a positive quantity changes in steps, where its value at each step is obtained from its value at the previous step by multiplying by the same scale factor, which is between 0 and 1 (exclusive). Then the quantity is said to **decay exponentially**.

If the starting number is a and the scale factor is b, then the value after n steps is given by the same formula as for exponential growth, namely ab^n.

Figure 6 shows the lengths of the pieces of paper at the beginning of the process and after each of the first ten steps. You can see that the decay is rapid at first, but gets slower and slower as the length gets closer to zero. This is typical of exponential decay.

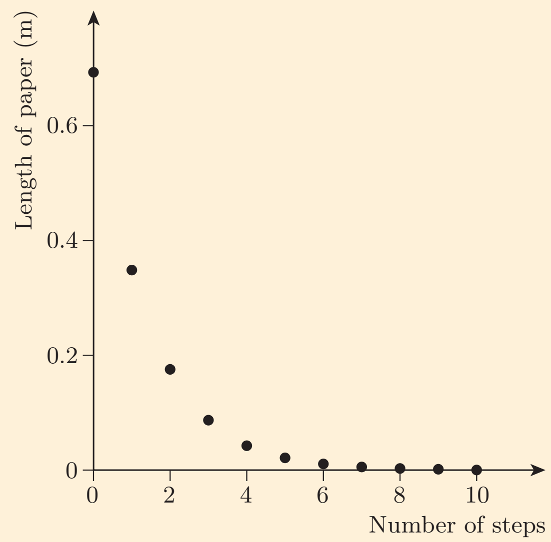


Figure 6 The length of the pieces of paper after each of the first ten steps

In the examples of exponential growth and decay that you have seen in this subsection, the scale factor was either 2 (corresponding to doubling) or $\frac{1}{2}$ (corresponding to halving). The scale factor in exponential change can be any positive number at all (except 1, which would give no change), and you will see some examples of exponential growth and decay with different scale factors in the next subsection.

1.2 Multiplying by a scale factor

In real-life examples of exponential growth and decay, scale factors are often given in the form of percentage increases or decreases. So this subsection starts by reminding you how to obtain the scale factor corresponding to a percentage increase or decrease. This idea is essential to your understanding of the rest of the unit.

Suppose, for example, that you want to increase the price of an item costing £18 by 15%. Since $100\% + 15\% = 115\%$, this means that the new price is 115% of the old price. So you can calculate the new price by multiplying the old price by the fraction $\frac{115}{100}$, which is equal to 1.15. The new price is

You saw how to work with percentage increases and decreases in Examples 14 and 15 on pages 41–42 of Unit 1.

$$£18 \times 1.15 = £20.70.$$

By the same principle:

to increase a number by 27%, multiply it by the scale factor 1.27;
to increase a number by 80%, multiply it by the scale factor 1.80, or simply 1.8;

and so on.

Care is needed with percentage increases of less than 10%. For example, if you want to increase a number by 6%, then you do *not* multiply it by the scale factor 1.6. You want to increase it to 106% of its original value, so you multiply by the fraction $\frac{106}{100}$, which is equal to 1.06.

The principle above also applies to percentage decreases. Suppose that you want to apply a 15% discount to the price of an item costing £18. Since $100\% - 15\% = 85\%$, the new price is 85% of the old price. So you can calculate the new price by multiplying the old price by the fraction $\frac{85}{100}$, which is equal to 0.85. The new price is

$$£18 \times 0.85 = £15.30.$$

By the same principle:

to decrease a number by 27%, multiply it by the scale factor 0.73 (since $100\% - 27\% = 73\%$);
to decrease a number by 80%, multiply by the scale factor 0.2 (since $100\% - 80\% = 20\%$);

and so on.

Scale factors for percentage increases and decreases

To increase a number by $r\%$, multiply it by $\dfrac{100+r}{100}$ $\left(= 1 + \dfrac{r}{100}\right)$.

To decrease a number by $r\%$, multiply it by $\dfrac{100-r}{100}$ $\left(= 1 - \dfrac{r}{100}\right)$.

You can practise converting percentage increases and decreases to scale factors in the next two activities.

Activity 3 *Finding the scale factors for percentage increases and decreases*

(a) A market stallholder buys leather purses for £4.25 each and applies a 65% mark-up to work out his selling price. By what scale factor must he multiply the buying price to work out the selling price? Use this scale factor to calculate the selling price.

(b) The same stallholder is offering a 30% discount on £7 shirts. By what scale factor must he multiply the usual price to work out the discounted price? Use this scale factor to work out the discounted price.

Activity 4 *More practice with scale factors*

(a) Write down the scale factor corresponding to each of the following percentage increases and decreases.

(i) 10% increase (ii) 3% increase (iii) 0.5% increase

(iv) 15% decrease (v) 2% decrease (vi) 1.5% decrease

(b) Write down the percentage increase or decrease corresponding to each of the following scale factors.

(i) 1.08 (ii) 0.91 (iii) 1.072

In each part of Activity 3, a number had to be multiplied by a scale factor only once. However, there are some practical situations where a number is repeatedly multiplied by a scale factor, leading to exponential growth or decay.

A common example is the growth in the value of a sum of money invested for a number of years at a fixed annual rate of interest. Usually the interest earned at the end of the first year is added to the initial investment, so in the second year interest is earned not just on the initial investment, but also on the amount of the first interest payment. Then the interest earned in the second year is also added to the value of the investment, and so on. The interest accrued in this way is called **compound interest**.

The value of the investment after one year is calculated by multiplying by the appropriate scale factor. The value of the investment after a greater number of years is calculated by repeatedly multiplying by the same scale factor, as illustrated in the example below.

Example 1 *Calculating the value of an investment*

A sum of £240 is invested in a deposit account that gives a 4% per year rate of return. There are no further transactions.

(a) How much will the investment be worth after the following times?

(i) 1 year (ii) 2 years (iii) 10 years

(b) Suppose that £V is the value of the investment after n years. Write down a formula for V in terms of n.

Solution

(a) (i) The rate of return is 4% per year, so by the end of 1 year the value of the investment will have increased by a scale factor of 1.04.

The value of the investment after 1 year is

$$£240 \times 1.04 = £249.60.$$

(ii) The value of the investment after 2 years is 1.04 times its value after 1 year.

The value after 2 years is

$$£240 \times 1.04 \times 1.04 = £240 \times 1.04^2$$
$$= £259.58 \text{ (to the nearest penny).}$$

(iii) The value after 10 years is

$$£240 \times 1.04 \times 1.04 \times 1.04 \times 1.04 \times 1.04$$
$$\times 1.04 \times 1.04 \times 1.04 \times 1.04 \times 1.04$$
$$= £240 \times 1.04^{10}$$
$$= £355.26 \text{ (to the nearest penny).}$$

(b) A formula for the value $£V$ of the investment after n years is

$$V = 240 \times 1.04^n.$$

Here are two activities involving compound interest for you to try.

Activity 5 *Working out compound interest*

Suppose that you invest £1800 at a fixed rate of 4.5% per year.

(a) How much will your investment be worth after the following times?

(i) 1 year (ii) 3 years (iii) 10 years

(b) Calculate the total interest earned since the money was invested, after each of the times in part (a).

(c) Suppose that $£V$ is the value of the investment, and $£W$ is the total amount of interest earned, after n years. Write down a formula for V in terms of n, and a formula for W in terms of n.

Activity 6 *Working out how much money to invest*

Suppose that you want to invest some money for a newborn baby so that she receives a gift of £1000 in 18 years' time. You have found an investment product that guarantees to pay compound interest of 5% per year for the whole of this period. You want to know how much money you must invest now in order to achieve your target.

(a) Suppose that the amount that you invest now is $£M$. Write down an expression for the value of the investment in 18 years' time, in terms of M.

(b) Hence write down an equation that M must satisfy if you are to achieve your target, and solve it to find how much money you must invest, to the nearest penny.

1.3 Discrete and continuous exponential growth and decay

In most of the examples of exponential growth and decay that you have seen in this unit so far, the change happened in steps, so the size of the quantity jumped from one value to the next, where each value was the previous value multiplied by the same scale factor. For example, the height of the pile of paper started at 0.0001 m, then it jumped to 0.0002 m (two times the first value), then to 0.0004 m (two times the value before), and so on.

However, consider the growth of the lily pads mentioned in Activity 2. The area of the pond covered by the lily pads doubled every day, so plotting the area covered after each day would give a graph like the one shown in Figure 7. Here n is the number of days and A is the area covered by the lily pads, measured in some suitable units, such as square metres.

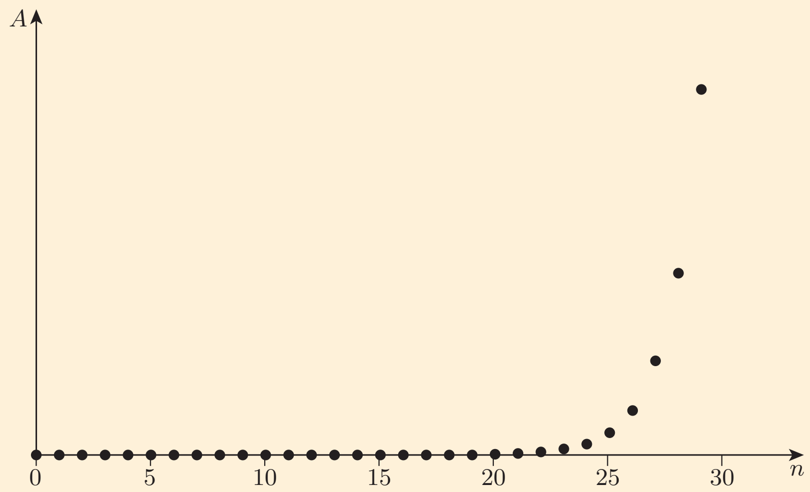


Figure 7 The area A covered by the lily pads after n days

As you know, each point on the graph in Figure 7 is given by a formula of the form

$$A = S \times 2^n,$$

where S is the area covered by the lily pads at the start of the 30 days, in the same units as A.

The area covered by the lily pads would not have grown in jumps, however – it would have grown continuously. The graph of its growth would have been something like the graph shown in Figure 8.

The curve in Figure 8 has exactly the same equation as the sequence of dots in Figure 7, namely

$$A = S \times 2^n. \tag{1}$$

The only difference is that the curve corresponds to the variable n taking *any* value, rather than just integer values. You saw in Unit 3 that you can raise a positive number to *any* power, not just powers that are integers.

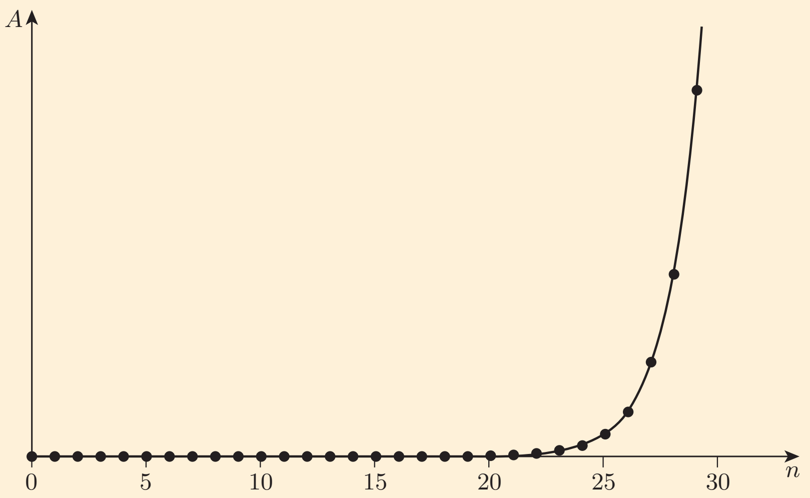


Figure 8 The area A covered by the lily pads after n days, as a continuous graph

You can use equation (1) to work out the area covered by the lily pads after $26\frac{1}{2}$ days, or 23.7 days, or any fractional number of days that you like (provided that you know the value of the initial area S): you substitute the appropriate number for n into the expression on the right-hand side of the equation and evaluate it. Your calculator will handle exponents that are not whole numbers in just the same way as whole-number exponents.

The definitions of exponential growth and decay that you saw in Subsection 1.1 can be extended to include the type of change illustrated in Figure 8, as follows.

Exponential growth and decay

A variable y is said to **change exponentially** with respect to a variable x if the relationship between x and y is given by an equation of the form

$$y = ab^x,$$

where a and b are positive constants, with b not equal to 1.

If $b > 1$, then y **grows exponentially**.

If $0 < b < 1$, then y **decays exponentially**.

If the change happens in steps (x takes values from a range of equally spaced numbers, such as the non-negative integers), then it is **discrete exponential change** (also called **geometric change**).

If the change happens continuously (x takes values from an interval of real numbers, such as the non-negative real numbers), then it is **continuous exponential change**.

The growth in the height of the paper pile is an example of *discrete* exponential growth. The height h metres of the pile is given by the formula

$$h = 0.0001 \times 2^n,$$

where n is the number of steps and takes the values $0, 1, 2, \ldots$.

In contrast, the growth in the area covered by the lily pads is an example

of *continuous* exponential growth. The area A is given by the formula

$$A = S \times 2^n,$$

where n is the time in days since the growth started, and takes any value that is a non-negative real number (perhaps up to some maximum value).

A curve like the one in Figure 8 – that is, a curve that is the graph of an equation of the form $y = ab^x$, where a and b are positive constants with b not equal to 1 – is called an **exponential curve**. If $b > 1$, as in Figure 8, then the curve is called an **exponential growth curve**. If $0 < b < 1$, then it is called an **exponential decay curve**.

Of course, the lily pad example is not very realistic, but many examples of real-life growth and decay can be modelled by exponential growth or decay curves. You will see some examples in the next two subsections.

1.4 Using continuous exponential models

Any model based on an equation of the form in the pink box on page 125 is called an **exponential model**. In particular, a model based on discrete exponential change is called a *discrete exponential model*, and similarly a model based on continuous exponential change is called a *continuous exponential model*. Usually you can choose a discrete or continuous exponential model according to which seems to best suit a situation.

An example of a situation that can be modelled by a continuous exponential model is the amount of a prescription drug in a patient's bloodstream. The concentration of the drug in the patient's bloodstream peaks shortly after it is administered, and then gradually falls as the drug is broken down or eliminated from the body. The concentration tends to drop steeply at first, but more slowly later, giving rise to a graph something like the one in Figure 9. This graph covers the period of time after the concentration of a particular drug in a patient's bloodstream peaks.

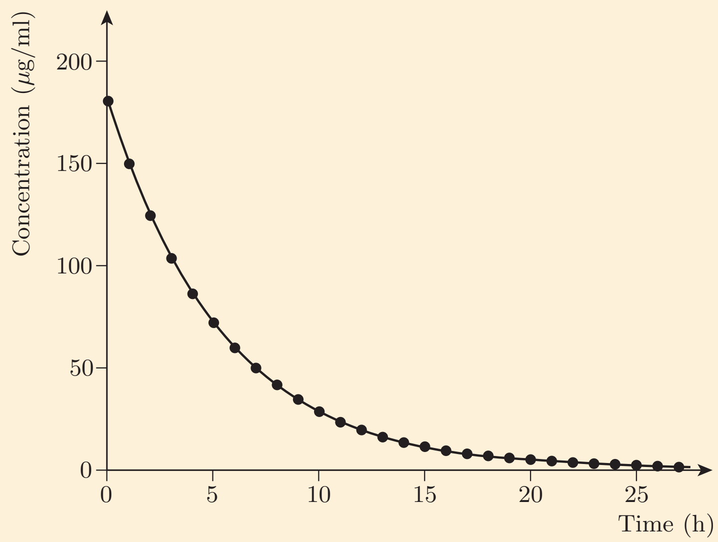


Figure 9 The concentration of a drug in a patient's bloodstream

A microgram (μg) is one millionth of a gram. The abbreviation mcg is sometimes used instead of μg.

In Figure 9, the peak concentration of the drug in the patient's bloodstream is 180 micrograms per millilitre (μg/ml). The curve models

the subsequent decline in the concentration, and the dots model the concentrations after whole numbers of hours. In this particular graph, the concentration represented by each dot is obtained from the concentration represented by the previous dot by multiplying by the scale factor 0.83, so the equation of the curve is

$$C = 180 \times 0.83^t,$$

where t is the time in hours and C is the concentration of the drug in the patient's bloodstream, in μg/ml.

Because the scale factor by which the concentration falls each hour is 0.83, and $83\% = 100\% - 17\%$, the percentage by which the concentration falls each hour is 17%.

It is important to remember that a graph like the one in Figure 9 is only a *model* for the decline in the concentration of a drug in a patient's bloodstream. The actual concentration at a particular time may not lie exactly on the curve.

As with all models, you should interpret any results obtained from an exponential model in an appropriate way. For example, you should not quote results too precisely, but should round them appropriately, as models give only approximate predictions. It is also important to make sure that the model used is appropriate for the situation. For example, the speed of the decline of a drug in a patient's bloodstream may vary from person to person.

In the next activity you are asked to use a model similar to the one discussed above to estimate the concentration of a drug in a patient's bloodstream after some different periods of time, including some that are not whole numbers of hours.

Activity 7 *Calculating the concentration of a drug in a patient's bloodstream*

A dose of a particular drug is administered to a patient, and the concentration of the drug in the patient's bloodstream peaks at 90 nanograms per millilitre (ng/ml) shortly afterwards. The concentration then decreases by 14% per hour. Let C ng/ml be the concentration t hours after the concentration peaked.

A nanogram (ng) is one billionth of a gram; that is, $1/10^9$ of a gram.

(a) Find the scale factor by which the concentration of the drug decreases every hour. Hence write down a formula for C in terms of t.

(b) Use your formula to find the expected concentration of the drug, to the nearest 5 ng/ml, at the following times after the concentration peaked.

(i) 4 hours (ii) 30 minutes

(c) Suppose that the peak concentration of the drug occurs 40 minutes after the drug was administered. Find the expected concentration of the drug in the patient's bloodstream two hours after it was administered, to the nearest 5 ng/ml.

Now suppose that you know the peak concentration of a drug in a patient's bloodstream, and the concentration one hour after this peak concentration. These two values can be estimated by using blood tests. You can use the two values to work out the scale factor by which the concentration of the drug decreased in the hour after the peak concentration was reached. If you assume that the concentration of the

drug will continue to decrease by the same scale factor every hour, then you can use this scale factor to write down a formula for the concentration of the drug any number of hours after the peak concentration.

For example, suppose that the peak concentration of a drug in a patient's bloodstream is $20\,\mu$g/ml, and the concentration after one hour is $14\,\mu$g/ml. Then over that hour the concentration of the drug changed by the scale factor

$$\frac{14}{20} = 0.7.$$

If the concentration of the drug continues to decrease by the same scale factor every hour, then the concentration $C\,\mu$g/ml at the time t hours after the concentration peaked is given by the formula

$$C = 20 \times 0.7^t.$$

Activity 8 *Finding a scale factor*

Suppose that the peak concentration of a drug in a patient's bloodstream is $36\,\mu$g/ml, and one hour later the concentration has dropped to $27\,\mu$g/ml. Assume that the concentration decreases by the same scale factor every hour, and let $C\,\mu$g/ml be the concentration t hours after the concentration peaked.

(a) Find the scale factor by which the concentration decreases each hour.

(b) Hence write down a formula for C in terms of t.

(c) Use your formula to work out the expected concentration 1 hour and 15 minutes after the peak concentration, to the nearest μg/ml.

As mentioned at the beginning of this subsection, you can usually choose a discrete exponential model or a continuous exponential model according to which seems to best suit a situation. You would choose a discrete exponential model if it is appropriate to model the quantity with change that takes place in steps *with the same scale factor at each step*. You would choose a continuous exponential model if it is appropriate to model the quantity with exponential change that happens continuously.

Even if a quantity does not change continuously, it is often appropriate to model it with a continuous exponential model. For example, suppose that the population of a new housing estate is increasing by about 15% every year. The size of population does not change continuously, as it jumps from one whole number of people to another whole number of people, rather than going through all the numbers in between. However, it is appropriate to model the size of the population with a continuous exponential model – as for any model, you would round any results obtained to a suitable degree of precision. Using a continuous exponential model allows you to estimate the size of the population at any time during the year, whereas using a discrete exponential model, based on steps at intervals of a year, allows you to estimate it only after whole numbers of years.

What you should remember is that the adjectives 'discrete' and 'continuous' refer not to the possible values taken by the quantity modelled, but to the values taken by the variable x in the equation $y = ab^x$ that is used in the model, as you will see if you look back at the pink box on page 125. (However, the adjective 'discrete' does not fully describe the nature of the values taken by the variable x in discrete exponential change, as they must be not just discrete, but *equally-spaced*.)

1.5 Exponential models from data

In Unit 6 you saw how to use Dataplotter to fit the best straight line to a set of data points. This process is called **linear regression**.

There is a similar process, called **exponential regression**, that can be used to fit the best exponential curve to a set of data points. Many computer spreadsheets and graphics calculators can carry out exponential regression (and linear regression). You will not be expected to carry out exponential regression in this module, but you should be aware that it can be done.

As an illustration, Table 1 gives some estimated data for the population of the world. Figure 10(a) shows a scatterplot of these data, and you can see that the points would be better modelled with a curve than with a straight line. Exponential curves frequently provide good models for the growth of populations – which can be of humans, animals or anything else – since populations often increase by a particular percentage every year. (For some types of population it can be appropriate to consider the percentage change over some other unit of time, instead of a year. For example, for a population of bacteria you might consider the percentage change every hour.)

Table 1 Estimated data for the population of the world

Year	Population
1850	1.260×10^9
1900	1.650×10^9
1920	1.860×10^9
1930	2.070×10^9
1950	2.520×10^9
1960	3.042×10^9
1990	5.282×10^9
2000	6.086×10^9
2009	6.786×10^9

Source: The first five data pairs come from United Nations (1999) *The World at Six Billion*; the remaining four data pairs come from the US Census Bureau.

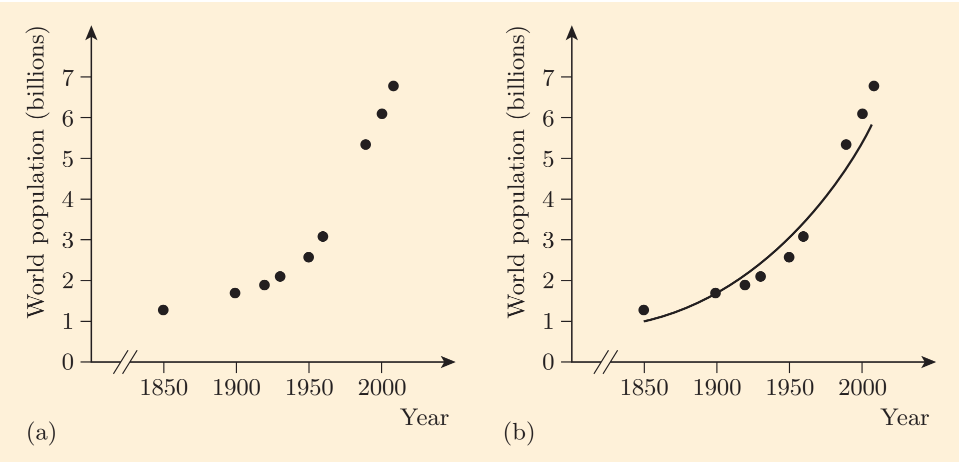


Figure 10 (a) A scatterplot of the data from Table 1. (b) The same scatterplot with the exponential regression curve superimposed.

Applying exponential regression to the data points in Table 1 shows that the exponential regression curve for these data has equation

$$y = 0.659\,543 \times 1.011\,47^x,$$

where x is the year, y is the size of the population, and the constants are given to six significant figures. This curve is shown in Figure 10(b). It is not a very close fit to the data points, but it is better than a straight line.

Activity 9 *Thinking about an exponential regression curve*

(a) What evidence is there from Figure 10(b) to show that, in recent years, the population of the world has been growing more quickly than the historical trend would suggest?

(b) Can you offer any possible explanation for this?

Figure 11 There are large flocks of ring-necked parakeets in south-east England.

If you look carefully at the scatterplot in Figure 10(a), you can see that if you were to draw a curve *exactly* through the points, it would get more and more steep until about 1990, when it would start to become slightly less steep. So it looks as if an exponential model for the population of the world is not appropriate in the long term. In fact, most exponential models are appropriate only for limited periods of time, as they eventually predict growth that is so rapid that it cannot happen in practice.

You need to be careful when you use an exponential model for a population that changes seasonally, as many animal populations do. For example, suppose that a survey has been carried out every September to estimate the size of a particular flock of ring-necked parakeets (Figure 11) in south-east England, and that the results of the survey are as shown on the graph in Figure 12(a). The equation of the exponential regression curve for these data points is

$$y = 10.5032 \times 1.284\,40^x,$$

where x is the time in years since 1 September 2000, y is the number of parakeets, and the constants are given to six significant figures.

This equation models the number of parakeets every September, but it is unlikely to model the number of parakeets at other times of the year. This is because the size of the flock would probably not follow a smooth curve, but might decline over the winter and then increase in the breeding season, perhaps as shown in Figure 12(b).

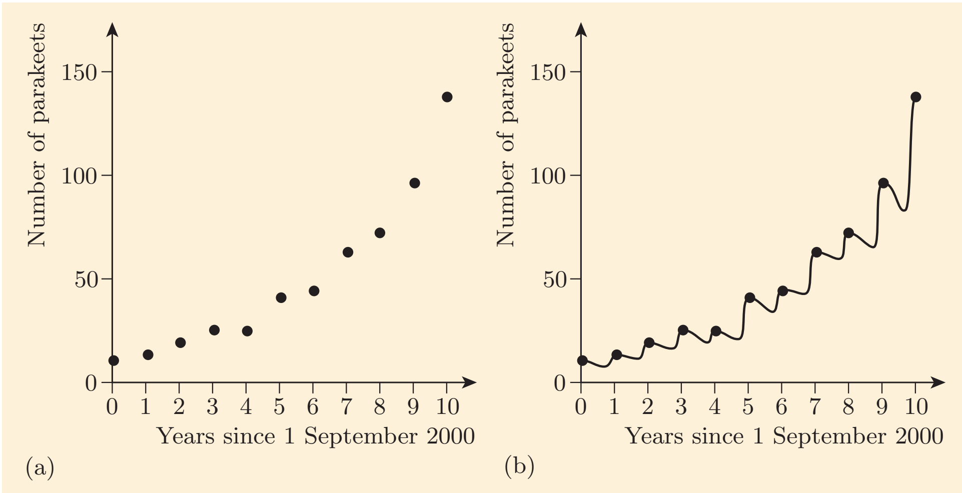


Figure 12 (a) A scatterplot showing the estimated number of parakeets every September from 1 September 2000. (b) A possible graph for the number of parakeets at all times of the year.

So the model is valid only for whole numbers of years after 1 September 2000 – that is, only for values of x that are non-negative integers. Because of this, it is appropriate to treat this model as a *discrete* exponential model, with the size of the flock changing in steps from September of one year to September of the next year.

For similar reasons to those for the flock of parakeets, if the lily pads discussed in Subsection 1.3 tend to grow more quickly at some times of the day than at others, then it might be more appropriate to model their growth by a discrete exponential model like that shown in Figure 7 on page 124, rather than by a continuous exponential model like that shown in Figure 8 on page 125.

As always with models, any results obtained from models like those in this subsection need to be interpreted appropriately. For example, the model for the number of parakeets predicts that after 11 years the number of parakeets in the flock will be 164.825.... This number should be rounded, as you cannot have a fractional number of parakeets! You could round it to 165, but since these sorts of models are often rather imprecise, a prediction of 'roughly 160' parakeets might be more appropriate.

In this section you have learned what it means for a quantity to grow or decay exponentially, and you have seen how to find and use formulas describing this type of change. You have also learned about the difference between discrete and continuous exponential change.

2 Working with exponential growth and decay

In this section you will see some further useful ways to work with exponential growth and decay.

2.1 Growth and decay to particular sizes

In the last section you used formulas for exponential growth and decay to work out answers to problems of the following type: How large will a quantity be after a particular time (or after a particular number of steps, in the case of discrete exponential growth)?

However sometimes you want to know the answer to a different type of problem: How much time (or how many steps) will it take until a quantity reaches a particular size?

For example, you might want to use an exponential model for the decline in the concentration of a drug in a patient's bloodstream to estimate the time when the concentration will fall to a particular level, so that a decision can be made about when the next dose of the drug should be administered. One way to make such an estimate is to obtain an accurate graph of the equation that describes the decline in the concentration (for example, using Graphplotter), and read the approprate value from the graph. You are asked to do this in the next activity.

Activity 10 *Predicting the time when a drug concentration will fall to a particular level*

Graphplotter

Suppose that the decline in the concentration of a drug in a patient's bloodstream is modelled by the equation

$$C = 30 \times 0.78^t,$$

where $C\,\mu$g/ml is the concentration t hours after the concentration peaked.

In parts (a)–(c) below you are asked to use Graphplotter to obtain the graph of this equation and estimate when the concentration will fall to a particular level.

(a) Open Graphplotter, and make sure that the 'One graph' tab is selected. Click the 'Options' tab and make sure that 'Grid', 'Axes' and 'Trace' are all selected, and all the other options are not selected. Click the 'Functions' tab to return to the main panel.

(b) Choose the equation $y = ab^x + c$ from the drop-down list. Set $a = 30$ and $b = 0.78$, and keep $c = 0$. Set the values of x min, x max, y min and y max to give the range 0 to 20 on the x-axis and the range 0 to 35 on the y-axis.

(c) Use the Trace facility to find the time, to the nearest half-hour, when the concentration falls to $10\,\mu$g/ml.

Figure 13 Typical advice to runners is to increase running distance by no more than 10% each week.

Another way to find the time, or number of steps, until a quantity that is changing exponentially reaches a particular level is to use **trial and improvement**. You start by guessing an answer and testing it, and then you repeatedly adjust your guess until you find the answer that you are looking for.

For example, consider the case of an athlete who has been running 20 km every week, but plans a new training schedule in which each week she will increase her running distance by 10% over the previous week. An increase of 10% corresponds to a scale factor of 1.1, so this means that in week 1 of her new schedule she will run a distance of $20 \times 1.1\,\text{km} = 22\,\text{km}$, in week 2 she will run $20 \times 1.1^2\,\text{km} = 24.2\,\text{km}$, and so on. In general, the distance in kilometres that she will run in week n is given by the expression

$$20 \times 1.1^n. \qquad (2)$$

The athlete wants to know the week of the schedule in which she will first be due to run more than 65 km, as at that point she plans to cease the schedule and stick to 65 km per week for a few weeks. The number of this week in the schedule is the smallest value of n for which

$$20 \times 1.1^n > 65.$$

The example below illustrates how you can use trial and improvement to find this value of n.

Example 2 *Using trial and improvement*

If the distance in kilometres run by an athlete in week n of a training schedule is given by expression (2), find the week number in which she is first due to run more than 65 km.

Solution

Guess a sensible value for the week number n, substitute it into the formula to calculate the corresponding running distance, then repeatedly adjust your guess until you find the answer. It can be helpful to set out the process in a table.

Guess for week number	Distance run in km (to the nearest 0.1 km)	Evaluation
20	$20 \times 1.1^{20} \approx 134.5$	Much too big
10	$20 \times 1.1^{10} \approx 51.9$	Too small
15	$20 \times 1.1^{15} \approx 83.5$	Too big
13	$20 \times 1.1^{13} \approx 69.0$	Still greater than 65
12	$20 \times 1.1^{12} \approx 62.8$	Less than 65

Since the athlete is due to run 62.8 km in week 12 and 69.0 km in week 13, the first week in which she is due to run more than 65 km is week 13.

The example above involves discrete exponential change, but you can also use trial and improvement in situations that are modelled by continuous exponential change, such as some changing populations. You are asked to try this in the next activity, which involves the number of Elvis Presley impersonators!

It has been reported that when Elvis Presley died on 16 August 1977, he had approximately 170 impersonators, but thirty years later, in 2007, the number had risen to roughly 85 000. In fact, it has been calculated that the number of Elvis impersonators is increasing by about 23% per year.

Figure 14 Is it the real Elvis, or one of his many impersonators?

Activity 11 *Calculating the time until everyone is an Elvis impersonator*

Suppose that the number of Elvis impersonators was 170 on 16 August 1977, and has been increasing by 23% per year since that date.

(a) Write down the scale factor by which the number of Elvis impersonators is increasing each year, and hence write down a formula for the number of Elvis impersonators t years after 16 August 1977.

(b) Check that your formula gives the answer 85 000, approximately, for the number of Elvis impersonators 30 years after 16 August 1977.

(c) Use trial and improvement to find, roughly (to within two years), the time that it would take for Elvis impersonators to account for the entire population of the world, if the population of the world remains at the size it is today, which is about 7 billion. Hence find, roughly, the calendar year in which you would expect this to happen.

The assumption that the population of the world will remain constant is of course unrealistic. There is more on this point in the text below the activity.

Clearly Activity 11 is only a bit of foolish fun, but it nevertheless illustrates a limitation of exponential models that was mentioned in the last section – they are often unlikely to be appropriate in the long term. Of course it won't happen that eventually the entire population of the world will be Elvis impersonators! In fact, it is probably unlikely that the annual growth rate of 23% in the number of Elvis impersonators will continue for many more years. So this exponential model for the growth in the number of Elvis impersonators might be realistic for a number of years, but it will not be a good model in the long term. Any exponential growth model eventually predicts growth so rapid that it cannot happen in practice.

You might also have noticed that the assumption in the question that the population of the world will remain the same is also unrealistic. In fact, as you saw in the last section, the population of the world is growing quite rapidly. At the time of writing, the population of the world is estimated to be increasing at a rate of about 1.13% per year. Although it may increase less rapidly in future, current predictions suggest that it could be as high as 9 billion in 2050. So even if the growth rate of the number of Elvis impersonators were to remain at 23% per year, if you want to compare the number of Elvis impersonators with the total number of people in the world, then you need to take into account the growth rate of the population of the world.

Trial and improvement can be a useful way of finding an answer to a mathematical problem when you don't know a better method. However, it can be time-consuming, especially if the answer that you are trying to find might be any value from an interval of real numbers (as opposed to just integer values, for example), and you want to find it fairly accurately. Later in the unit you will learn a quicker method that can be used to answer questions of the type considered in this subsection. It involves the idea of *logarithms*.

2.2 Growth and decay over different lengths of time

In this subsection you will learn about a useful property of exponential growth and decay.

As an illustration, consider again the paper pile discussed in Subsection 1.1. Its height doubles at each step, so if you know the height after a particular step, and you want to know the height after one more step, then you multiply by 2. Similarly, if you know the height after a particular step, and you want to know the height after *two* more steps, then you multiply by 2 twice – that is, you multiply by 2^2. And to work out the height of the pile from its height *three* steps earlier, you multiply by 2^3, and so on. In general, to work out the height of the pile from its height i steps earlier, you multiply by 2^i. In other words, every i steps the height of the paper pile changes by the scale factor 2^i.

You can see that the fact in the box below is true for discrete exponential change in general.

Discrete exponential change over different numbers of steps

Suppose that a quantity changes by the scale factor b at each step.

Then every i steps it changes by the scale factor b^i.

Annual percentage rates (APRs), which you have probably seen quoted in relation to loans such as credit card debts, are worked out using this fact. Suppose that you have a debt on which you are charged interest at 2% per month, and you do not repay any of the money owing. After each month the interest is added to the debt, and then the interest for the next month is calculated on the total amount owing. That is, the interest is *compounded* each month.

You met the idea of compound interest in Subsection 1.2, although there the discussion was from the point of view of an investor, whereas here it is from the point of view of a debtor.

In the case of the debt discussed above, the compound interest is 2% per month, so your debt increases by the scale factor 1.02 each month. Over a year (12 months) your debt is increased by this scale factor 12 times, so, by the property in the box above, altogether your debt increases by the scale factor

$$1.02^{12} \approx 1.27.$$

In other words, the annual interest rate that you are being charged is about 27%. This number, 27%, is the *annual percentage rate*, or APR.

Notice that the APR of 27% is greater than 12 times the monthly rate of 2%. This is because the interest is compounded. This example illustrates why it is important to consider the APR when you pay interest, rather than just the monthly rate, for example. Small monthly rates of interest may lead you to believe that you are getting a good deal, whereas in fact the annual rate of interest can be higher than you think.

APRs provide the best way to compare loan offers from lenders who charge interest in different ways, such as every year, or every six months. For example, suppose that you were offered the opportunity to transfer the debt discussed above to a different lender, at 26% annual interest. This is a slightly better option, since the 2% monthly rate charged by the current lender is equivalent to an annual rate of 27%.

Here is another example that illustrates how to calculate an APR.

Example 3 *Calculating an APR*

Calculate the APR for an interest rate of 4.5%, charged quarterly (that is, every three months), giving your answer as a percentage to one decimal place.

Solution

An interest rate of 4.5% charged quarterly corresponds to a scale factor of 1.045 quarterly. This gives a scale factor of

$$1.045^4 = 1.193 \text{ (to 3 d.p.)}$$

per year, which gives an interest rate of 19.3% (to 1 d.p.) per year. That is, the APR is approximately 19.3%.

Activity 12 *Calculating APRs*

(a) Calculate the APR for each of the following interest rates, as a percentage to one decimal place.

(i) An interest rate of 17.5%, charged annually

(ii) An interest rate of 8.5%, charged every six months

(iii) An interest rate of 1.4%, charged monthly

(b) If you have three loan offers, with the three interest rates in part (a), which would you choose?

There is a version of the property in the pink box on page 134 that is useful when you are dealing with *continuous* exponential change.

To illustrate it, let's consider a particular continuous exponential model. The value of a car tends to drop more in the early years of its existence than in the later years, and the decline in the value can often be modelled by a continuous exponential model. The decline in the value of a car (or any item) is known as **depreciation**.

For example, the exponential curve in Figure 15 (overleaf) models the value of a car whose value at purchase is £20 000 and which depreciates by 20% per year; that is, by a scale factor of 0.8 per year. The equation of the curve is

$$v = 20\,000 \times 0.8^t,$$

where v is the value of the car in £ after t years. The dots on the curve model the values of the car after whole numbers of years.

In practice, graphs like the one in Figure 15 are usually appropriate models only for cars that are at least a year old, as the value of a new car tends to drop by a much higher percentage in its first year than in subsequent years.

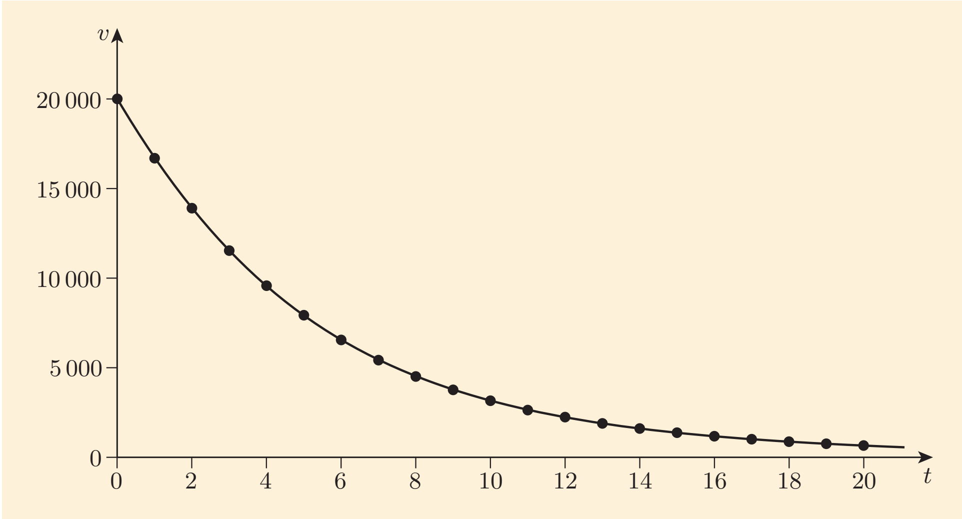


Figure 15 The graph of the equation $v = 20\,000 \times 0.8^t$. The variable t represents the number of years, and v is the value of the car in £.

You know that the value of the car at each dot is 0.8 times its value at the previous dot. So, from what you saw earlier in this subsection, if you know the value of the car after any whole number of years, and you want to work out its value i years later, where i is any whole number, then you multiply by 0.8^i.

But what if you know the value of the car at some point in time, not necessarily after a whole number of years, and you want to work out its value after some further length of time, which is again not necessarily a whole number of years? In fact, you can work out the new value in exactly the same way as above – if the length of time is 1.5 years, say, then you just multiply the old value by $0.8^{1.5}$. This even works for negative periods of time: if you want to know the value of the car 2.25 years, say, *before* a particular time, then you multiply by $0.8^{-2.25}$. The reason why this works is explained shortly, but first here is an example to illustrate how you can apply this idea in practice.

Figure 16 Values of classic cars, like this Volkswagon Beetle, tend to depreciate more slowly than those of other cars, and may even *appreciate* (increase).

Figure 17 A Volkswagon Beetle whose value has certainly depreciated significantly!

Example 4 *Working out some values of a car*

A car depreciates at the rate of 17% per year. Its value three and a half years after purchase was £10 920. How much, to the nearest ten pounds, was it worth at the following times?

(a) 1.5 years later (b) At purchase

Solution

The car depreciates at the rate of 17% per year, so its values change by the scale factor 0.83 each year.

(a) 1.5 years after the value of the car was £10 920, its new value was

$$£10\,920 \times 0.83^{1.5} = £8260 \text{ (to the nearest £10).}$$

(b) The value of the car at purchase was

$$£10\,920 \times 0.83^{-3.5} = £20\,960 \text{ (to the nearest £10).}$$

Here is a similar activity for you to try.

Activity 13 *Working out some values of a car*

A car depreciates at the rate of 16% per year. Its value one year after purchase was £15 450. How much, to the nearest ten pounds, was it worth at the following times?

(a) 0.25 years later (b) 2.75 years *after purchase* (c) At purchase

The useful fact used in Example 4 and Activity 13 can be summarised as follows.

Continuous exponential change over different time periods

Suppose that a quantity is subject to continuous exponential change by the scale factor b every year.

Then over any time interval of length i years, it changes by the scale factor b^i.

(The same is true if time is measured in any other unit, such as weeks, days or minutes.)

To see why this is true, consider, for example, the car mentioned on page 135, whose depreciation is modelled by the equation

$$v = 20\,000 \times 0.8^t,$$

where v is the value of the car in £ after t years. Consider two points in time that are i years apart, say t years after purchase and $t + i$ years after purchase. The values in £ of the car at these two times are

$$20\,000 \times 0.8^t \quad \text{and} \quad 20\,000 \times 0.8^{t+i},$$

respectively. By an index law that you met in Unit 3, the second value of the car can be rewritten as

$$20\,000 \times 0.8^t \times 0.8^i,$$

so you can see that it is 0.8^i times the first value. Exactly the same argument will hold whatever the annual scale factor and starting value are.

The index law used here is
$$a^m \times a^n = a^{m+n}.$$

So if a quantity is subject to continuous exponential change over time, then it doesn't have just one scale factor associated with its change: *different lengths of time correspond to different scale factors.* For example, if a quantity changes by the scale factor 1.3 every day, then

over any week (7 days) it changes by the scale factor $1.3^7 \approx 6.27$,

over any hour ($\frac{1}{24}$ day) it changes by the scale factor $1.3^{1/24} \approx 1.01$,

and so on.

Activity 14 *Working out scale factors over different lengths of time*

A population of pigeons in a city is thought to be increasing at a rate of 80% per decade.

(a) By what scale factor does the number of pigeons grow each decade?

(b) By what scale factor does the number of pigeons grow each year? Give your answer to four significant figures.

(c) By what percentage increase does the number of pigeons grow each year? Give your answer to one decimal place.

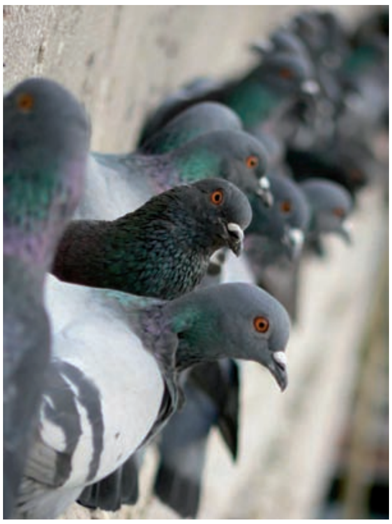


Figure 18 Urban pigeons

The fact in the pink box on page 137 is true even if the unit of time over which the initial scale factor is measured is not a standard one: it could be 4 weeks, or 12.7 years, and so on. For example, if a quantity changes by the scale factor b every 4 weeks, then over any time interval of length 10 weeks (which is 2.5 times 4 weeks) the quantity changes by the scale factor $b^{2.5}$.

In this section you have seen how to use either a graphical method or trial and improvement to calculate the length of time (or number of steps) that a quantity that is changing exponentially will take to reach a particular value. You will see a better method, involving logarithms, for solving problems like this in Section 5. You have also seen how to calculate scale factors for exponential change over different lengths of time (or over different numbers of steps).

3 Exponential curves

In this section you will explore the graphs of functions with rules of the form

$$y = ab^x,$$

You learned about the meaning of b^x, where b and x are not necessarily integers, in Unit 3.

where a and b are constants, with b positive. The constant b must be positive because if b is negative then b^x has no meaning for non-integer values of x, and if b is zero then b^x has no meaning for non-positive values of x. As you saw in Subsection 1.3, a function whose rule is of the form above, with $a > 0$ and $b \neq 1$, corresponds to exponential change from the starting value a by the scale factor b, and its graph is called an *exponential curve*.

We start by looking at a more restricted family of functions: those with rules of the form

$$y = b^x,$$

where b is a positive constant. A function of this form with $b \neq 1$ is called an **exponential function**. It corresponds to exponential change from the starting value 1, since its rule is of the more general form $y = ab^x$ with $a = 1$.

In some texts, the term 'exponential function' has a less restricted definition than that given here. For example, the term might be used to refer to any function with a rule of the form $y = ab^x$, where a and b are constants, with b positive.

3.1 Graphs of equations of the form $y = b^x$

This subsection is about the graphs of exponential functions. Before you explore these graphs, let's first compare a particular exponential function with two functions of other types. Here are the rules of three different functions, each of which can be used to model a positive quantity y that increases as a positive quantity x increases:

$$y = 2x, \quad y = x^2, \quad y = 2^x.$$

These three functions are commonly confused with each other, so it is important that you are able to distinguish between them. In the next activity you are asked to compare how fast the y-values of these three functions increase.

Activity 15 *Comparing growth models*

(a) Look at the three functions at the bottom of the opposite page. State whether each is linear, quadratic or exponential.

(b) The table below shows some of the y-values of each function for $x = 0, 1, 2, 3, 4, 5, 6$. Calculate the missing y-values and write them in the table. Then compare how the y-values of each function increase as x increases from 0 to 6. Which function has the greatest increase, and which has the smallest?

x	0	1	2	3	4	5	6
$y = 2x$	0		4		8	10	
$y = x^2$	0		4		16	25	
$y = 2^x$	1		4		16	32	

In Activity 15 you should have found that the growth of $y = 2^x$ outstripped the growth of $y = x^2$, which itself outstripped the growth of $y = 2x$. In fact, the growth of any exponential function $y = b^x$ with $b > 1$ will always eventually outstrip the growth of any quadratic function $y = ax^2 + bx + c$ with $a > 0$, which itself will always eventually outstrip the growth of any linear function.

The exponential function that you looked at in Activity 15 was the one with the simple rule $y = 2^x$. To explore the graphs of exponential functions, let's start with this function.

The exponential function $y = 2^x$

We will plot a graph of the equation $y = 2^x$ by constructing a table of values. The values of y for some positive values of x have already been found in Activity 15, so let's now find the values of y for some negative values of x.

You could do this by using your calculator, but so that you can understand why the answers are as they are, let's instead use the definition of what it means to raise a number to a negative power, which you saw in Unit 3. The definition is summarised by the index law

$$a^{-n} = \frac{1}{a^n}.$$

This gives

$$2^{-1} = \frac{1}{2^1} = \tfrac{1}{2} = 0.5,$$

$$2^{-2} = \frac{1}{2^2} = \tfrac{1}{4} = 0.25,$$

$$2^{-3} = \frac{1}{2^3} = \tfrac{1}{8} = 0.125,$$

$$2^{-4} = \frac{1}{2^4} = \tfrac{1}{16} = 0.0625,$$

and so on. If you think about how this pattern continues, you'll see that the values will get smaller and smaller, but they will never reach zero.

Table 2 shows the values of y for integer values of x between -4 and 4.

Table 2 A table of values for the equation $y = 2^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	0.0625	0.125	0.25	0.5	1	2	4	8	16

The values of 2^x where x is not an integer lie on a smooth curve between the values of 2^x where x is an integer. You learned the meaning of these values in Unit 3. For example,

$$2^{\frac{1}{2}} = \sqrt{2}, \quad 2^{\frac{3}{2}} = \left(\sqrt{2}\right)^3 \quad \text{and} \quad 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}.$$

If you plot the nine points given by Table 2 and join them with a smooth curve, then you obtain the graph in Figure 19. It is an exponential *growth* curve, as you would expect, because the equation $y = 2^x$ corresponds to exponential change (from the starting value 1) by the scale factor 2, which is greater than 1.

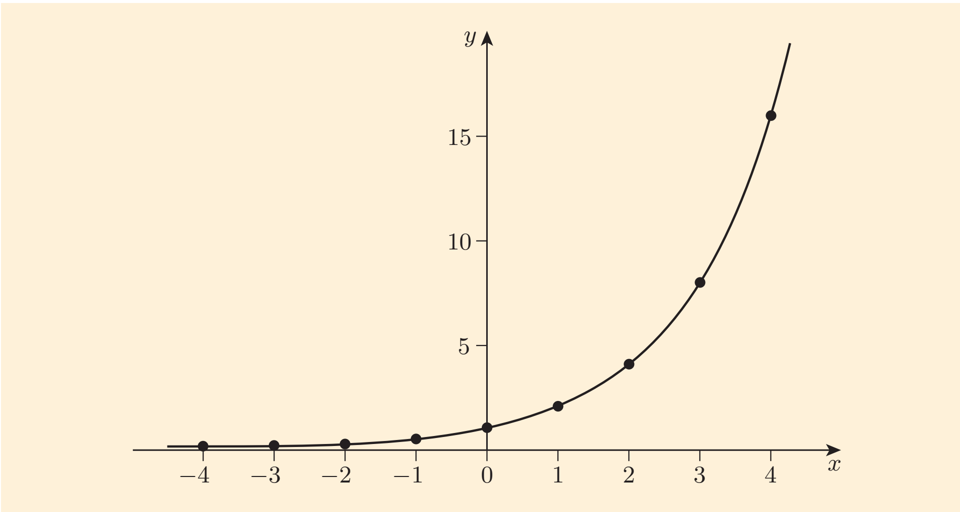


Figure 19 The graph of $y = 2^x$

There are several features worth noticing about this graph. First, the y-intercept is 1. This is because substituting $x = 0$ into the equation $y = 2^x$ gives $y = 2^0$, and you saw in Unit 3 that raising any number to the power 0 gives 1.

The heights of the dots on and to the right of the y-axis are the values of 2^x when $x = 0, 1, 2$ and so on. These values start at 1, when $x = 0$, and then repeatedly double, so, as you would expect, they increase rapidly.

The heights of the dots to the left of the y-axis are the values of 2^x when $x = -1, -2, -3$, and so on. As you have seen, these values get smaller and smaller as x decreases, but never reach zero.

The graph of $y = 2^x$ lies entirely above the x-axis. Although it never reaches the x-axis, it does get closer and closer to it. This behaviour is described by saying that the x-axis is an *asymptote* of the graph.

You met the word 'asymptote' in Unit 12, in connection with the graph of the tangent function. You saw that the tangent function has infinitely many asymptotes, each of which is a vertical line. An asymptote can be a horizontal or vertical line, or even a slant line.

Now let's consider the graph of another simple exponential function.

The exponential function $y = \left(\frac{1}{2}\right)^x$

To plot the graph of the equation $y = \left(\frac{1}{2}\right)^x$, we begin by constructing a table of values. You can find the values of y for negative values of x by using the index laws

$$a^{-1} = \frac{1}{a} \quad \text{and} \quad (a^m)^n = a^{mn}. \qquad (3)$$

The first of these two index laws is a special case of the index law

$$a^{-n} = \frac{1}{a^n}.$$

These give

$$\left(\tfrac{1}{2}\right)^{-1} = \left(2^{-1}\right)^{-1} = 2^1 = 2,$$
$$\left(\tfrac{1}{2}\right)^{-2} = \left(2^{-1}\right)^{-2} = 2^2 = 4,$$
$$\left(\tfrac{1}{2}\right)^{-3} = \left(2^{-1}\right)^{-3} = 2^3 = 8,$$
$$\left(\tfrac{1}{2}\right)^{-4} = \left(2^{-1}\right)^{-4} = 2^4 = 16,$$

and so on. You can see that these values will grow rapidly.

Table 3 shows the values of y for integer values of x between -4 and 4.

Table 3 A table of values for the equation $y = \left(\frac{1}{2}\right)^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	16	8	4	2	1	0.5	0.25	0.125	0.0625

Notice that the y-values in Table 3 are just the same as the y-values in Table 2, but in reverse order. This is because for any number x,

$$\left(\tfrac{1}{2}\right)^x = (2^{-1})^x = 2^{-x},$$

by index laws (3) again.

The values of $\left(\frac{1}{2}\right)^x$ where x is not an integer lie on a smooth curve between the values of $\left(\frac{1}{2}\right)^x$ where x is an integer.

If you plot the nine points given by Table 3 and join them with a smooth curve, then you obtain the graph in Figure 20. It is an exponential *decay* curve, as you would expect, because the equation $y = \left(\frac{1}{2}\right)^x$ corresponds to exponential change (from the starting value 1) by the scale factor $\frac{1}{2}$, which is between 0 and 1.

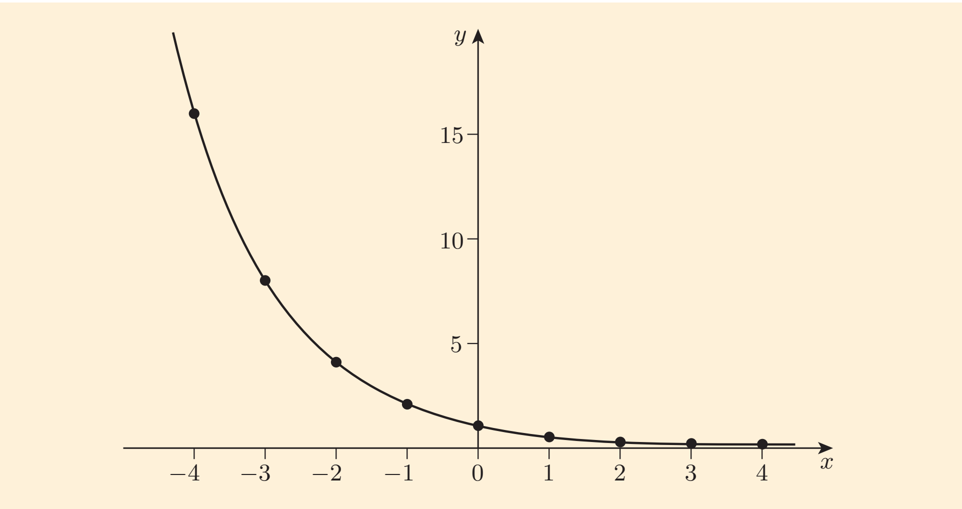


Figure 20 The graph of $y = \left(\frac{1}{2}\right)^x$

There are several features worth noticing about this graph, too. Like the

graph of $y = 2^x$, the graph of $y = \left(\frac{1}{2}\right)^x$ has y-intercept 1. The reason is the same: raising any number to the power 0 gives 1, so, in particular, $\left(\frac{1}{2}\right)^0 = 1$.

The heights of the dots on and to the right of the y-axis are the values of $\left(\frac{1}{2}\right)^x$ when $x = 0, 1, 2$, and so on. These values start at 1, when $x = 0$, and then repeatedly halve, so they get smaller and smaller, but never reach zero.

The heights of the dots to the left of the y-axis are the values of $\left(\frac{1}{2}\right)^x$ when $x = -1, -2, -3$, and so on. As you have seen, these values grow rapidly as x decreases.

Like the graph of $y = 2^x$, the graph of $y = \left(\frac{1}{2}\right)^x$ lies entirely above the x-axis, and has the x-axis as an asymptote. As x increases, the values of $y = \left(\frac{1}{2}\right)^x$ get closer and closer to the x-axis but never reach it.

Other exponential functions

You have now looked in detail at the graphs of the equations $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$. In the next activity you are asked to use Graphplotter to investigate the graphs of some other exponential functions.

Graphplotter

Activity 16 *Investigating graphs of equations of the form $y = b^x$*

Use Graphplotter, with the 'One graph' tab selected. Click the 'Autoscale' button to ensure that the axis scales are at their default values. Tick the y-intercept checkbox on the Options page, and untick the Trace checkbox, if it is ticked.

(a) Choose the equation $y = ab^x + c$ from the drop-down list. Make sure that $a = 1$ and $c = 0$, and keep a and c set to these values throughout the activity, since the aim is to explore the graphs of equations of the form $y = b^x$.

(b) Make sure that $b = 2$, and check that you have obtained the graph of $y = 2^x$, as expected.

(c) Vary the value of b (remembering that it must be *positive*) and write down what you discover about the effect that this has on the graph. In particular, try setting $b = 1$.

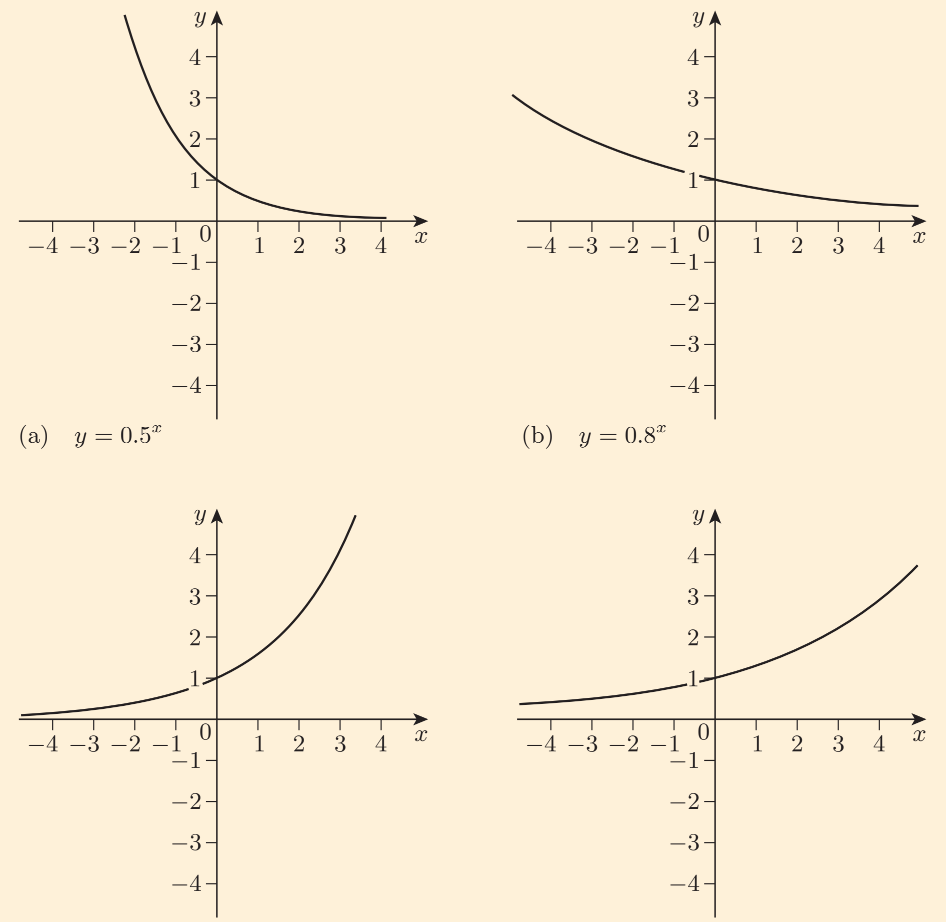
(d) What do you notice about the y-intercept of the graph for different values of b? Can you explain why this happens?

You should have found in Activity 16 that in both the range $0 < b < 1$ (which gives decay curves) and the range $b > 1$ (which gives growth curves), the closer the value of b is to 1, the flatter is the graph. This makes sense, because b is the scale factor, so the closer it is to 1, the less change you get when you multiply a number by b, so the slower is the growth or decay. You should have seen that when $b = 1$ the graph is completely flat!

You should also have observed that all the graphs in Activity 16 have y-intercept 1: the reason for this is given in the solution to Activity 16. You might have noticed, too, that all of the graphs lie entirely above the x-axis, and all of them, except the straight-line graph obtained by putting $b = 1$, have the x-axis as an asymptote.

You can see some of these features of the graphs of exponential functions in the four graphs in Figure 21. Graphs (a) and (b) illustrate $0 < b < 1$

(decay curves) and graphs (c) and (d) illustrate $b > 1$ (growth curves). You can see that the values of b closer to 1 give the flatter graphs. Notice also that all four graphs have y-intercept 1 and the x-axis as an asymptote.



(a) $y = 0.5^x$ (b) $y = 0.8^x$ (c) $y = 1.6^x$ (d) $y = 1.3^x$

Figure 21 Graphs of equations of the form $y = b^x$

3.2 Graphs of equations of the form $y = ab^x$

In this subsection you will look at the graphs of equations of the form $y = ab^x$, where b is positive.

Graphplotter

Activity 17 *Investigating graphs of equations of the form $y = ab^x$*

Use Graphplotter, with the 'One graph' tab selected. Tick the y-intercept checkbox on the Options page, if it is not already ticked.

(a) Choose the equation $y = ab^x + c$ from the drop-down list, if it is not already selected. Make sure that $c = 0$, and keep c set to 0 throughout this activity, since the aim is to explore the graphs of equations of the form $y = ab^x$.

(b) Set $b = 2$ and vary the value of a. Write down what you discover about the effect that this has on the graph. Look in particular at the y-intercept.

(c) Repeat part (b) for one or two other (positive) values of b.

In Activity 17, you should have found that, regardless of the value of b, the y-intercept is equal to the value of a. This is because substituting $x = 0$ in the equation $y = ab^x$ gives $y = ab^0 = a \times 1 = a$. Some examples of graphs of equations of the form $y = ab^x$ are shown in Figure 22, and you can see that in each case the y-intercept is the value of a.

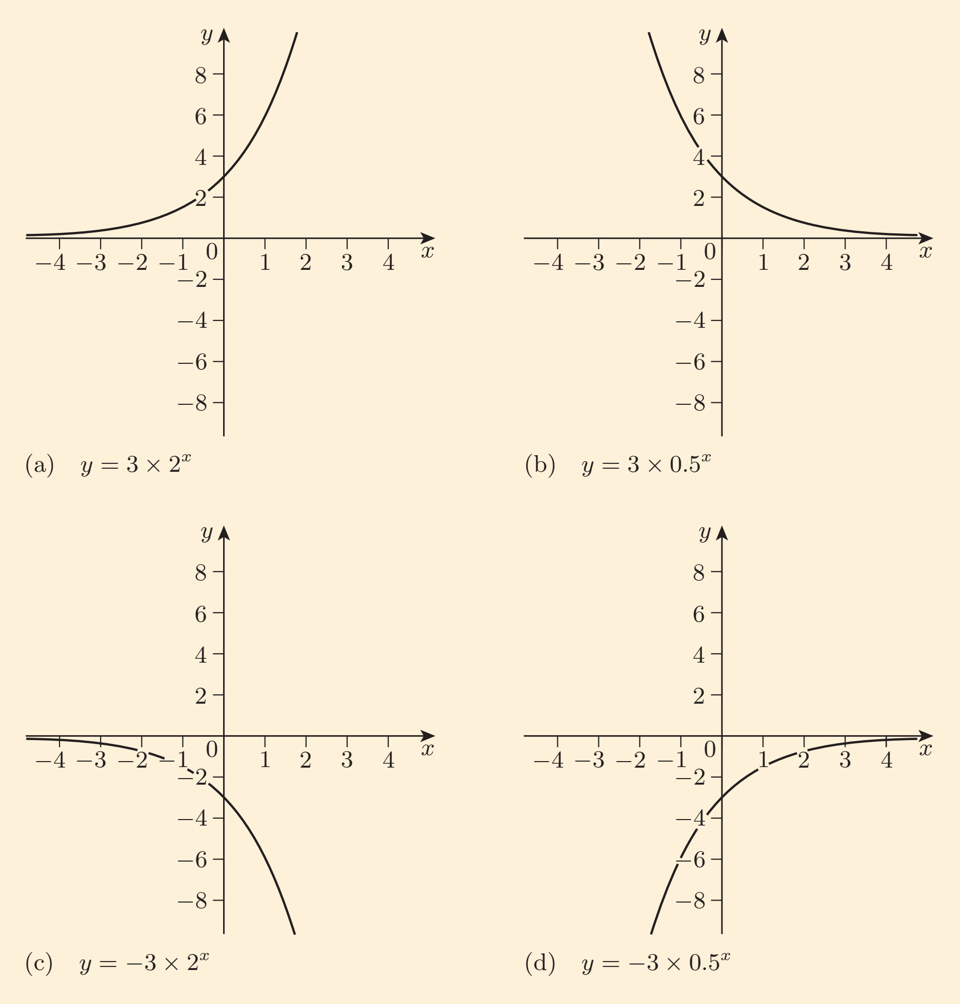


Figure 22 Graphs of equations of the form $y = ab^x$

You should also have found in Activity 17 that positive values of a produce exponential growth and decay curves (provided that $b \neq 1$). This is as you would expect, because if a is positive and $b \neq 1$ then the equation $y = ab^x$ corresponds to exponential change by the scale factor b from the starting value a.

You should have found that the larger the value of a, the steeper the curve. This is because the graph of $y = ab^x$ is obtained from the graph of $y = b^x$ by multiplying each y-value by a, which moves the corresponding points on the curve up or down vertically. The larger the size of the y-value before it is multiplied by a, the more the corresponding point moves, which causes the steepness of the curve to increase.

You saw other examples of this mirror-image effect in Unit 10, when you considered what happens to a parabola with an equation of the form $y = ax^2$ when you change the value a to its negative.

Negative values of a produce curves of a type that you have not seen so far. The shapes of these curves are the shapes of exponential growth or decay curves reflected in the x-axis. To see why this happens, think about changing the value of a in the equation $y = ab^x$ to its negative. This changes all the y-values to their negatives, and the effect is that the graph is changed to its mirror image, reflected in the x-axis. You might like to use the ‘Two graphs’ tab in Graphplotter to explore pairs of graphs of the

form $y = ab^x$, where the two graphs have the same value of b and values of a that are negatives of each other.

It is important to appreciate that reflecting an exponential growth curve in the x-axis does not give the shape of an exponential decay curve, and similarly that reflecting an exponential decay curve in the x-axis does not give the shape of an exponential growth curve. So curves with equations of the form $y = ab^x$ where a is negative are *not* exponential growth or decay curves. For example, Figure 22(a) shows an exponential growth curve, and Figure 22(c) shows its reflection in the x-axis. Although the graph in Figure 22(c) is decreasing, it becomes *steeper* as you run your eye from left to right, whereas an exponential decay curve, like that in Figure 22(b), becomes *flatter*.

You should have found that for negative values of a, as well as for positive values, the larger the magnitude of a, the steeper the curve.

Finally, you might have noticed that if a is positive then the graph of $y = ab^x$ lies entirely above the x-axis, while if a is negative then the graph lies entirely below the x-axis. All the graphs, except those produced by putting $b = 1$, have the x-axis as an asymptote.

Activity 18 *Choosing values of a and b that give exponential decay*

Without looking at any of the graphs in this unit, write down a value of a and a value of b such that the graph of $y = ab^x$ is an exponential decay curve.

You will have noticed that Graphplotter allows you to explore functions of the form $y = ab^x + c$. From your experience of graphs in earlier units, you should know that the graph of $y = ab^x + c$ will be the same shape as the graph of $y = ab^x$, but shifted vertically by the amount c. The shift is upwards if c is positive and downwards if c is negative. This happens because adding the constant c to the right-hand side of the equation $y = ab^x$ just changes all the y-values by c units, which causes the graph to move up or down. You might like to try this on Graphplotter.

So, for example, since the graph of $y = ab^x$ has the x-axis as an asymptote, the graph of $y = ab^x + c$ has the horizontal line with equation $y = c$ as an asymptote (provided that $a \neq 0$ and $b \neq 1$).

Here is a summary of what you have learned about the graphs of functions of the form $y = ab^x$ in the last two subsections.

Graphs of equations of the form $y = ab^x$

- If $a > 0$ then the graph lies entirely above the x-axis. If also

$$\begin{array}{ll} b > 1, & \text{then the graph is an exponential growth curve;} \\ 0 < b < 1, & \text{then the graph is an exponential decay curve;} \\ b = 1, & \text{then the graph is a horizontal line.} \end{array}$$

- If $a < 0$ then the graph lies entirely below the x-axis and is neither an exponential growth curve nor an exponential decay curve.
- The x-axis is an asymptote (except when $a = 0$ or $b = 1$).
- The y-intercept is a.
- The closer the value of b is to 1 (and the closer the value of a is to 0) the flatter is the graph.

3.3 Euler's number e

In Subsection 3.1 you investigated the graphs of exponential functions, which are functions with rules of the form

$$y = b^x,$$

where b is positive. You saw that all the graphs go through the point $(0, 1)$. You also saw that if $b > 1$ then the graph is an exponential growth curve, and the larger the value of b, the steeper the graph. Some examples are shown in Figure 23.

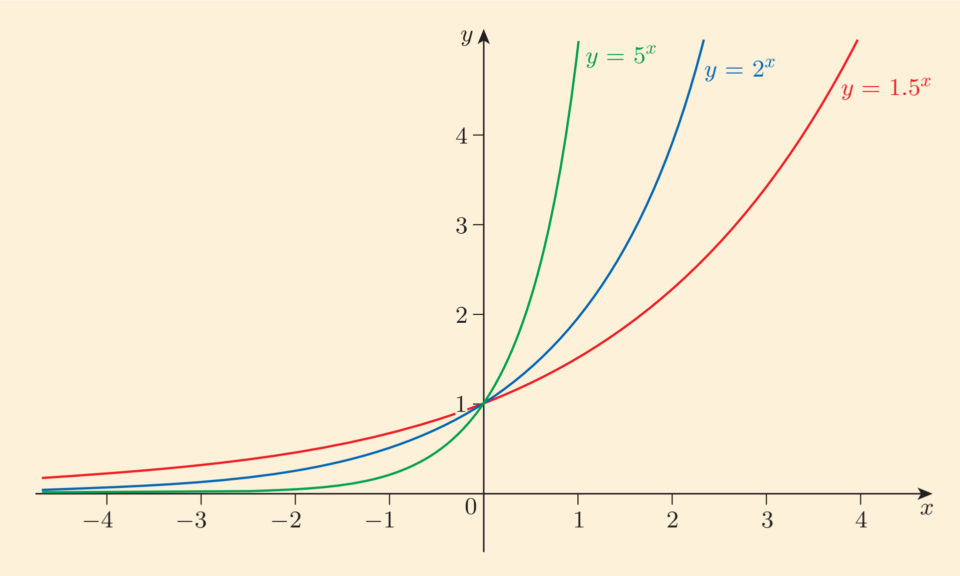


Figure 23 The graphs of three exponential functions

In the next activity you are asked to investigate the steepness of the graphs of exponential functions at the point $(0, 1)$.

Graphplotter

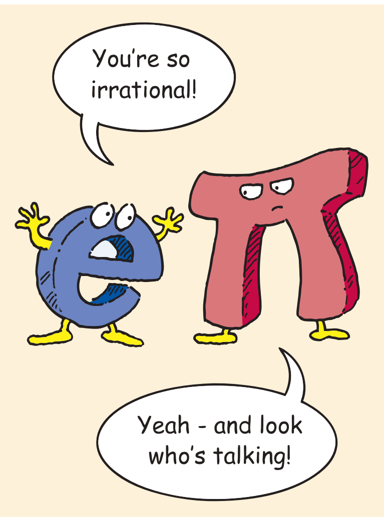
Activity 19 *Investigating the gradient of the graph of $y = b^x$ at $(0, 1)$*

Use Graphplotter, with the 'One graph' tab selected. Choose the equation $y = ab^x + c$ from the drop-down list, if it is not already selected.

(a) Set $a = 1$ and $c = 0$, and keep them set to these values, since this activity is about equations of the form $y = b^x$. Set $b = 2$, to begin with.

(b) Zoom in on the point $(0, 1)$, by setting x min and x max to be -0.01 and 0.01, respectively, and y min and y max to be 0.99 and 1.01, respectively. At this level of detail the graph looks like a straight line, so you can get a good idea of how steep it is.

(c) Now use the slider to change the value of b until the gradient of the graph appears to be exactly 1; that is, until the graph goes up by the same distance vertically as it goes along horizontally. What value of b seems to achieve this?



In Activity 19 you should have found that the value of b that gives a gradient of 1 at $(0, 1)$ seems to be about 2.7. In fact, the precise value is a special number sometimes called **Euler's number**, whose first few digits are 2.718 28.... Like π, Euler's number is irrational, so its digits have no repeating pattern and it cannot be written down exactly as a fraction or terminating decimal. It is usually denoted by the letter e.

So the exponential function with the rule $y = e^x$ has the special property that its gradient is exactly 1 at the point $(0, 1)$. If you choose to study mathematics at a level beyond MU123, then you will learn about other, related, properties of this particular exponential function.

This function is important both in applications of mathematics and in pure mathematics, and because of its importance it is sometimes referred to as ***the* exponential function**. Its rule is sometimes written as $y = \exp x$, instead of $y = e^x$. The value of e is available from your calculator keypad, just as for π, and you can also work out values of e^x by using a function button on your calculator, as you will see later in the unit.

At higher levels of mathematics, the two mathematical constants e and π crop up everywhere, in all sorts of contexts.

There is a document on the module website that shows you another context, related to exponential growth, in which the constant e arises naturally.

The use of the letter e for the number $2.718\,28\ldots$ was introduced by the Swiss mathematician Leonhard Euler (Figure 24). He first used it in a manuscript that he wrote in 1727 or 1728, which described some experiments involving the firing of a cannon. The first appearance of the notation e in a published work was in Euler's *Mechanica* (1736), a fundamental book on mechanics.

It is not known why Euler chose the letter 'e': it could just have been because it is the first vowel after 'a', which Euler was already using in his work. (It was probably not because 'e' is the first letter of 'exponential' or 'Euler'.) Over the next two decades Euler went on to make various important discoveries about the number e.

Euler was a hugely prolific mathematician who made fundamental discoveries in many areas of mathematics, including number theory, calculus and, as mentioned above, mechanics. At present his collected works run to 73 volumes, with many volumes of his scientific correspondence and other manuscripts yet to appear. He was responsible for bringing into general use many pieces of mathematical notation that are still used today, including, as you saw in Unit 8, the notation π for the ratio of the circumference of a circle to its diameter.

'Euler' is pronounced 'oiler'.

Figure 24 Leonhard Euler (1707–1783)

In this section you have explored the shapes of the graphs of equations of the form $y = ab^x$, where a and b are constants with b positive. You have also met the number e, otherwise known as Euler's number, which has value $2.718\,28\ldots$.

4 Logarithms

In Activity 11 on page 133 you were asked to find roughly how many years after 16 August 1977 it would take for Elvis impersonators to account for the entire population of the world, assuming that there were 170 Elvis impersonators on that date, their number grows by 23% per year, and the population of the world remains constant at 7 billion. Under these assumptions, the number of Elvis impersonators after t years is

$$170 \times 1.23^t,$$

so the time t in years that it would take for everyone in the world to be an

Elvis impersonator is the solution of the equation

$$170 \times 1.23^t = 7\,000\,000\,000.$$

Equations like this, in which the unknown is in an exponent, are known as **exponential equations**. It is often useful to solve exponential equations when you are working with exponential models.

You were asked in Activity 11 to find an approximate solution of the equation above by using trial and improvement, and you also saw in Subsection 2.1 that another way to find approximate solutions to equations like this is to use graphs. However, there is a quicker method for solving exponential equations. It involves the use of *logarithms*, which you will learn about in this section. Logarithms are often called *logs* for short.

Logarithms are useful not only for solving equations, but also in many other contexts, such as in *logarithmic scales*, as you will see. It is important to have a good understanding of logarithms if you plan to take further mathematics modules.

4.1 A brief history of logarithms

Figure 25 John Napier (1550–1617)

The invention of logarithms is attributed to the sixteenth-century Scottish mathematician John Napier (Figure 25). One of the major hindrances to the mathematical work needed to create astronomical and navigational tables at that time was the difficulty of multiplying and dividing large numbers. Napier bemoaned the 'tedious expense of time' involved in these calculations and the fact that they were 'subject to many slippery errors'. So he tried to find a way of making the calculations easier.

Napier's breakthrough came about by studying the properties of numbers raised to a power. To take a simple example, he considered what happened when he multiplied, say, 2^3 and 2^5. Using the rule that you saw in Unit 3 for multiplying numbers in index form, $a^m \times a^n = a^{m+n}$, he obtained

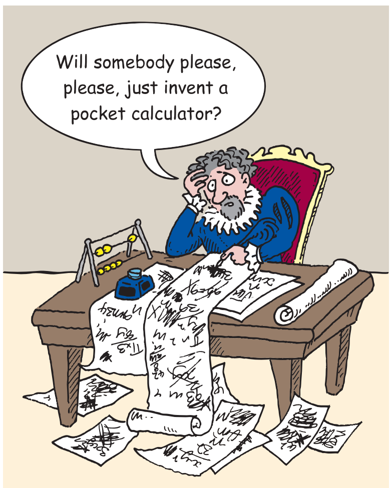
$$2^3 \times 2^5 = 2^{3+5} = 2^8.$$

Recall from Unit 3 that the number b in an expression of the form b^x is called the *base* or *base number*. The number x is called the *power*, *index* or *exponent*.

He observed that what started off as the problem of *multiplying* two numbers, 2^3 and 2^5, had been reduced to the much simpler problem of *adding* two numbers, 3 and 5. The key to this trick was that the numbers to be multiplied were expressed as powers of the same *base* – in this case the base is 2.

This idea can be applied to any multiplication. A particular number must be chosen as the base – any positive number except 1 can be used, but the obvious choice is the number 10. So, in order to multiply, say, 287 and 37, you would rewrite each of these numbers as ten to some power (approximately), and proceed like this:

$$\begin{aligned}287 \times 37 &\approx 10^{2.4579} \times 10^{1.5682}\\ &= 10^{2.4579+1.5682}\\ &= 10^{4.0261}\\ &\approx 10\,619.\end{aligned}$$



This makes the multiplication easier, because it is done by doing an addition instead. But there remains the problem of finding the correct powers of ten for the numbers to be multiplied, and of turning the power of ten obtained as the answer back into an ordinary number.

This problem was solved in the early seventeenth century by drawing up sets of tables to be used whenever a calculation like this was to be performed. The tables gave the power of ten corresponding to each number that might need to be multiplied – these are called the **logarithms** of the numbers. The tables could also be used to find the ordinary number corresponding to each power, which are called the **antilogarithms** of the powers. Figure 26 shows an extract from an eighteenth-century book of such tables (the title page of this book was shown in Unit 12, on page 63), which shows that the logarithm of 37 is 1.568 2017, to seven decimal places. In other words,

$$37 \approx 10^{1.568\,2017}.$$

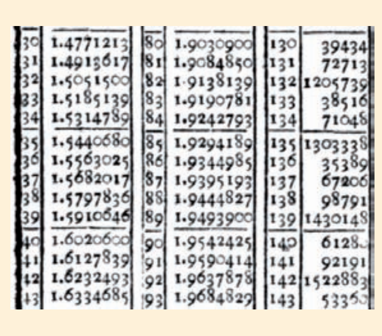
So, to multiply any two numbers, the procedure would be to use the tables to find the corresponding logarithms, add these logarithms together, and then use the tables again to find the antilogarithm of the result.

The precision of the answer obtained by using the logarithm method depends on the precision of the logarithms used. The seven decimal places in the old table in Figure 26 were sufficient for most practical purposes.

The logarithm method can be used for division as well as multiplication. To divide one number by another, you *subtract* their logarithms. For example,

$$\begin{aligned} 287 \div 37 &\approx 10^{2.4579} \div 10^{1.5682} \\ &= 10^{2.4579-1.5682} \\ &= 10^{0.8897} \\ &\approx 7.757. \end{aligned}$$

(This calculation uses the rule for dividing numbers in index form that you met in Unit 3: $a^m/a^n = a^{m-n}$.)



30	1.4771213	80	1.9030900	130	39434
31	1.4913617	81	1.9084850	131	72713
32	1.5051500	82	1.9138139	132	1205739
33	1.5185139	83	1.9190781	133	38516
34	1.5314789	84	1.9242793	134	71048
35	1.5440680	85	1.9294189	135	1303338
36	1.5563025	86	1.9344985	136	35389
37	1.5682017	87	1.9395193	137	67206
38	1.5797836	88	1.9444827	138	98791
39	1.5910646	89	1.9493900	139	1430148
40	1.6020600	90	1.9542425	140	6128
41	1.6127839	91	1.9590414	141	92191
42	1.6232493	92	1.9637878	142	1522883
43	1.6334685	93	1.9684829	143	5336

Figure 26 An extract from a book of logarithm tables published in 1717

The historical procedure described here was used to multiply numbers with more significant figures than 287 and 37. (The numbers 287 and 37 can be multiplied fairly quickly by using long multiplication, but are used here to illustrate the ideas.)

John Napier came from a distinguished Scottish family and held the title of the eighth Laird of Merchiston. He was educated at the University of St Andrews, which he entered in 1563 at the age of 13, and he also studied elsewhere in Europe. He spent most of his time running his estates in Scotland and working on theology – he was a fervent Protestant. Mathematics was only a hobby for him.

Napier's first work on logarithms was his *Mirifici logarithmorum canonis descriptio*, which was published in Latin in 1614. Part of this work consisted of tables listing the logarithms of the sines of angles, which could be used to simplify the work of ships' navigators. The East India Company, an English company that pursued trade with Eastern lands, had the *Descriptio* translated into English for use by its seafarers.

For several hundred years after Napier's invention, logarithm tables and slide rules (Figure 27), which exploit the same principle, were essential aids to calculation. They were finally swept away only with the arrival of cheap pocket calculators in the late 1970s. Logarithms themselves still have many other uses, as you will see in this section. Subsection 4.2 will help you to become familiar with the idea of logarithms and how to find them, and in the later subsections and the next section you will learn about the various ways in which they are useful.

Figure 27 A slide rule

The invention of logarithms made things much easier for astronomers, whose work in calculating the orbits of planets involved a great deal of arithmetic. Two hundred years after their invention, the French scientist Pierre-Simon Laplace (1749–1827) said that logarithms 'by shortening the labours, doubled the life of the astronomer'.

4.2 Logarithms to base 10

In the previous subsection you met the idea that the logarithm of a number is the power to which 10 must be raised to give the number. For example, you saw that the logarithm of 37 is approximately 1.5682, because

$$10^{1.5682} \approx 37.$$

Similarly, the logarithm of 100 is exactly 2, because

$$10^2 = 100.$$

Logarithms like these are more accurately called **logarithms to base 10**, because their values depend on the fact that the number 10 has been chosen as the base. You will see examples of logarithms with other bases later in the section, but in this subsection you will concentrate on logarithms to base 10, which are also known as **common logarithms**.

Napier coined the word 'logarithm' from the Greek words *logos* ('proportion, ratio') and *arithmos* ('number'). The 'arithm' part of the word comes from the same root as the start of 'arithmetic'.

Figure 28 shows how the statement that the logarithm to base 10 of 100 is 2 is written in mathematical notation.

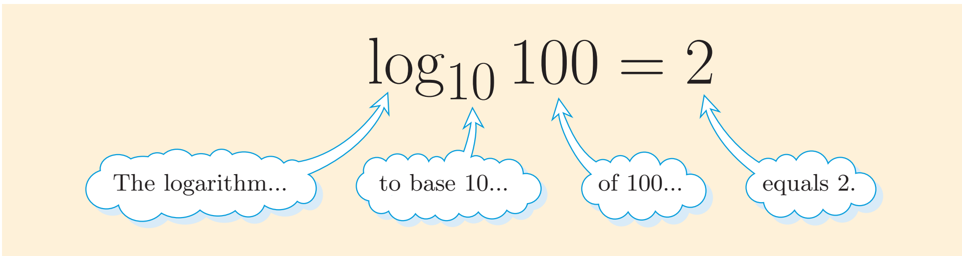


Figure 28 Notation for logarithms

So the two statements

$$10^2 = 100 \quad \text{and} \quad \log_{10} 100 = 2$$

are equivalent to each other – they mean the same thing.

You can use your calculator to find logarithms to base 10, and you will have a chance to practise this shortly. First, however, to make sure that you have a good understanding of the relationship between numbers and their logarithms, let's look at some examples of logarithms to base 10 that you can find without using your calculator.

You have already seen that the logarithm to base 10 of 100 is 2, since the power to which 10 must be raised to give 100 is 2. In general, if you can express a number as a power of 10, then you can write down its logarithm immediately, since it is just the power. For example, you can write 10 000 as 10^4, so the logarithm to base 10 of 10 000 is 4. That is,

$$\log_{10} 10\,000 = 4.$$

You can find the logarithms of any of the numbers 10, 100, 1000, 10 000, and so on, in this way, without using your calculator.

You can also express the numbers 0.1, 0.001, 0.0001, and so on, as powers of 10, and hence write down their logarithms, without using your calculator. For example, the number 0.01 has its 1 in the hundredths place, so

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}.$$

The last step in the manipulation here follows from the index law

$$a^{-n} = \frac{1}{a^n}.$$

Hence the logarithm to base 10 of 0.01 is -2. That is,

$$\log_{10} 0.01 = -2.$$

Notice that if you know the logarithm of a number, then you can work out the number simply by finding the appropriate power of 10. For example, if you know that the logarithm of a number is -4, then the number is

$$10^{-4} = \frac{1}{10^4} = \frac{1}{10\,000} = 0.0001.$$

Here are some more examples.

Tutorial clip

Example 5 *Understanding logarithms*

Do the following without using your calculator.

(a) Find the logarithms, to base 10, of the following numbers.

(i) A million (ii) 1 (iii) 0.000 01

(b) Find the two consecutive integers between which the number $\log_{10} 568$ lies. Do the same for the number $\log_{10} 0.037$.

(c) The logarithms to base 10 of two numbers are 4 and -1. What are the numbers?

Solution

(a) The logarithm to base 10 of a number is the power to which 10 must be raised to give the number. So start by expressing each of the given numbers as a power of 10.

(i) A million is 10^6, so $\log_{10}(1 \text{ million}) = 6$.

(ii) $1 = 10^0$, so $\log_{10} 1 = 0$.

(iii) $0.000\,01 = \dfrac{1}{100\,000} = \dfrac{1}{10^5} = 10^{-5}$, so $\log_{10} 0.000\,01 = -5$.

(b) Find two powers of 10, with integer exponents, between which the number 568 lies.

The number 568 lies between 100 and 1000, that is, between 10^2 and 10^3. The logarithms to base 10 of 10^2 and 10^3 are 2 and 3, respectively, so $\log_{10} 568$ lies between 2 and 3.

Find two powers of 10, with integer exponents, between which the number 0.037 lies.

The number 0.037 lies between 0.01 and 0.1, that is, between 10^{-2} and 10^{-1}. The logarithms to base 10 of 10^{-2} and 10^{-1} are -2 and -1, respectively, so $\log_{10} 0.037$ lies between -2 and -1.

(c) If the logarithm to base 10 of a number is 4, then the number is $10^4 = 10\,000$.

Similarly, if the logarithm to base 10 of a number is -1, then the number is $10^{-1} = 0.1$.

Here is a similar activity for you to try.

Activity 20 *Exploring logarithms*

Do the following without using your calculator.

(a) Find the following logarithms.

(i) $\log_{10} 10$ (ii) $\log_{10} 1000$ (iii) $\log_{10} (1 \text{ billion})$

(iv) $\log_{10} 0.001$

(b) Does the number zero have a logarithm to base 10?

(c) In each of part (c)(i) to (iv), find two consecutive integers between which the logarithm lies.

(i) $\log_{10} 91$ (ii) $\log_{10} 2971$ (iii) $\log_{10} 8.8$ (iv) $\log_{10} 0.25$

(d) The logarithms to base 10 of two numbers are 5 and -2. What are the numbers?

You saw in Activity 20 that the number zero does not have a logarithm to base 10. In fact, only *positive* numbers have logarithms. For example, the number -1 doesn't have a logarithm to base 10, because there is no power to which 10 can be raised to give -1.

Notice, however, that logarithms *themselves* can be negative. For example, you saw in Example 5 that the logarithm to base 10 of 0.000 01 is -5.

So, in summary, *only positive numbers have logarithms*, but *logarithms themselves can be any number*.

The box below summarises the definition of a logarithm to base 10.

Logarithms to base 10

The **logarithm to base 10** of a number x, denoted by $\log_{10} x$, is the power to which you have to raise 10 to get the answer x. So the two equations

$$x = 10^y \quad \text{and} \quad y = \log_{10} x$$

are equivalent.

Sometimes, when it is understood that a logarithm is a common logarithm, the subscript 10 is omitted from the logarithm notation. So, for example, we might write simply

$$\log 100 = 2,$$

rather than

$$\log_{10} 100 = 2.$$

It is also common to omit the subscript when the notation is used in an equation involving logarithms that is true no matter what the base is.

Later in the unit you will see examples of equations involving logarithms that are true no matter what the base is. (The base must be the same for each logarithm in the equation.)

You have seen that if a number is written as a power of 10, then you can write down its logarithm to base 10 immediately. If a number is not written as a power of 10, then it is not straightforward to find its logarithm to base 10, but you can use your calculator to obtain an approximate value. The next activity shows you how to do this.

Activity 21 *Logarithms on your calculator*

Work through Subsection 3.10 of the MU123 Guide.

In mathematics there are many pairs of operations that 'undo' or 'reverse the effect of' each other. For example, finding the cube of a number and finding the cube root of a number are like this. If you start with 5, say, and cube it, then you get 125; and if you then take the cube root, you get 5 again. Or you can apply the operations in the other order: for example, if you start with 8 and take the cube root then you get 2, and if you then cube this number you get 8 again.

Two operations that undo the effect of each other are called **inverse operations**.

Raising 10 to a number and finding the logarithm of a number to base 10 are inverse operations. For example, if you start with the number 5 and raise 10 to this number then you get $10^5 = 100\,000$; and if you then take the logarithm to base 10, you get 5 again. Or you can do the operations in the other order: for example, if you start with 100 and take the logarithm to base 10 then you get 2, and if you then raise 10 to this number, you get 100 again.

The inverse operation to finding the logarithm to base 10 of a number, which is raising 10 to a number, was called finding the *antilogarithm* in Subsection 4.1. This word is not commonly used nowadays, since the demise of logarithm tables!

You might like to try some more examples on your calculator: choose any (reasonably small) number and raise 10 to that number, then take the logarithm to base 10 of the result, and check that you get the number that you started with. Try applying the two operations in the other order, as well.

Raising 10 to a large power will give a number that is too big for your calculator to handle.

4.3 Logarithmic scales

Whenever you hear news of an earthquake reported in the media, its magnitude on the *Richter scale* is always mentioned. Table 4 shows the numbers of earthquakes between 2000 and 2008 measuring 5 or more on the Richter scale.

Table 4 The numbers of earthquakes by magnitude between 2000 and 2008 (data from United States Geological Survey Earthquake Hazards Program)

Year	5.0 to 5.9	6.0 to 6.9	7.0 to 7.9	8.0 to 9.9	Total 5.0 to 9.9
2003	1203	140	14	1	1358
2004	1515	141	14	2	1672
2005	1693	140	10	1	1844
2006	1712	142	9	2	1865
2007	1995	177	14	4	2190
2008	530	73	6	0	609

Source: http://earthquake.usgs.gov

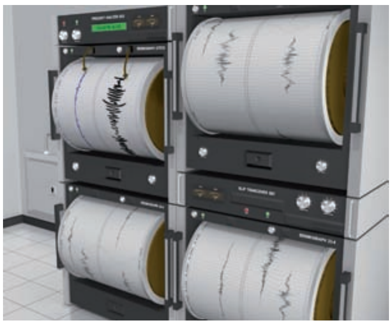


Figure 29 *Seismographs* measure and record ground motion

The Richter scale measures the amount of seismic energy released by an earthquake. It is not a normal, 'linear' scale, but a **base 10 logarithmic scale**. What this means is that the size of an earthquake of magnitude 8,

say, on the Richter scale should be thought of as 10^8, rather than simply the number 8. So, for example, an earthquake of magnitude 9 on the Richter scale is ten times more powerful than one of magnitude 8, because 10^9 is ten times larger than 10^8. Similarly, an earthquake of magnitude 6 on the Richter scale is a hundred times more powerful than one of magnitude 4, because 10^6 is a hundred times larger than 10^4. The magnitude of an earthquake on the Richter scale can be thought of as the logarithm to base 10 of the amount of energy released.

The intensity of an earthquake varies with the distance from its centre, so scales other than the Richter scale are used to describe the *local* intensity of earthquakes. These scales are often based on observations of damage to buildings – an example is the European Macroseismic Scale.

Logarithmic scales are useful for measuring quantities where the numbers vary from very small to extremely large, like the amount of seismic energy released by an earthquake. However, scales of this type need to be interpreted with care, as small differences on a logarithmic scale can represent very large differences in the quantities being measured.

Another well-known example of a base 10 logarithmic scale is the *decibel* (dB) scale, which is most commonly used for measuring the intensity of sound. This scale is slightly more complicated than the Richter scale, because the fundamental unit is not the decibel but the *bel* – a decibel is one tenth of a bel. A sound level of 9 bels is 10 times as strong as one of 8 bels.

Activity 22 *Comparing values on logarithmic scales*

(a) An earthquake was recorded as having a magnitude of 6.6 on the Richter scale. A few minutes later, an aftershock was measured at 4.6. How did the amount of energy released by the two quakes compare?

(b) If the noise level in an industrial workplace goes up from 60 decibels (6 bels) to 70 decibels (7 bels), by what factor does the intensity of the sound increase?

4.4 Logarithms to other bases

Now let's consider logarithms to bases other than 10. Here is the general definition of a logarithm, to any base b.

Logarithms

The **logarithm to base** b of a number x, denoted by $\log_b x$, is the power to which you have to raise the base b to get the answer x. So the two equations

$$x = b^y \quad \text{and} \quad y = \log_b x$$

are equivalent.

(Remember that:

- the base b must be positive and not equal to 1;
- only positive numbers have logarithms, but logarithms themselves can be any number.)

The number 1 cannot be the base of logarithms because raising 1 to a power always gives the value 1. The reason why only positive numbers have logarithms is explained for base 10 on page 152; a similar reason applies for any base.

As with logarithms to base 10, if you can express a number as a power of the base b, then you can immediately write down its logarithm to base b.

Example 6 *Understanding logarithms to base 2*

Find the logarithms to base 2 of the following numbers.

(a) 16 (b) $\frac{1}{8}$

Solution

You need to find the powers to which you have to raise the base 2 to get the given numbers. So start by expressing each of the numbers as a power of 2.

(a) $16 = 2^4$, so $\log_2 16 = 4$.

(b) $\frac{1}{8} = \dfrac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

The manipulation in part (b) uses the index law

$$a^{-n} = \frac{1}{a^n}.$$

Here is a similar activity for you to try.

Activity 23 *Exploring logarithms to base 2*

Find the logarithms to base 2 of the following numbers, without using your calculator.

(a) 8 (b) 32 (c) 2 (d) 1 (e) 0.5 (f) $\sqrt{2}$

You have seen that finding the logarithm of a number to base 10 and raising 10 to a number are inverse operations. The analogous fact holds for any base b: finding the logarithm of a number to base b and raising b to a number are inverse operations.

Euler's number e is often used as a base for logarithms in higher-level mathematics, because of the importance of the exponential function with rule $y = e^x$, and the fact that finding a logarithm to base e is the inverse operation to raising e to a number. Logarithms to base e turn out to be easier to work with in some ways than logarithms to any other base.

Logarithms to base e are called **natural logarithms**. The notation 'ln' is usually used in place of '$\log_e$', so, for example, the natural logarithm of 5 is usually written as

$$\ln 5$$

rather than $\log_e 5$. The box below summarises the definition of a natural logarithm, using this notation.

Logarithms to base e were first described as 'natural' logarithms in the seventeenth century, by the Danish mathematician Nicolaus Mercator (1620–1687).

Natural logarithms

The **natural logarithm** of a number x, denoted by $\ln x$, is the power to which you have to raise the base e to get the answer x. So the two equations

$$x = e^y \quad \text{and} \quad y = \ln x$$

are equivalent.

In some disciplines the natural logarithm of x is denoted by $\log x$, rather than $\ln x$ or $\log_e x$. You saw earlier that $\log x$ is also often used to denote $\log_{10} x$, or alternatively the logarithm of x with no specific base (but the same base for each use of the notation), so if you see this notation used outside this module then it is worth checking its meaning.

As with logarithms to other bases, if a number is expressed as a power of e, then it is straightforward to write down its logarithm to base e, as illustrated in the next example.

The first published use of the notation ln for natural logarithm was in a book written by an American mathematician, Irving Stringham, which was published in 1893. He explained his choice, in a textbook published a little later, as follows: 'In place of ${}^{e}\log$ we shall henceforth use the shorter symbol ln, made up of the initial letters of *logarithm* and of *natural* or *Napierian*.'

Example 7 *Understanding natural logarithms*

Find the natural logarithms of the following numbers.

(a) e^4 (b) $e^{-3.27}$ (c) e

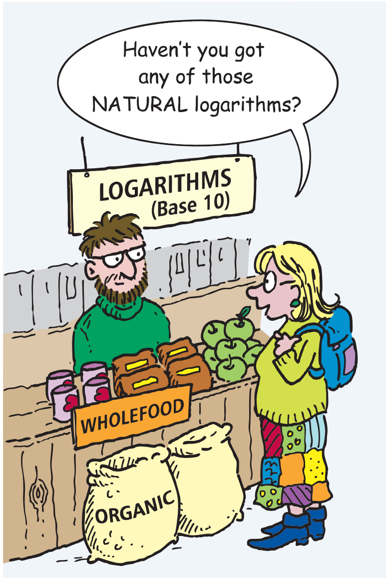
Solution

The power to which e must be raised to give e^4 is 4, so the natural logarithm of e^4 is 4. This gives the answer to part (a), and the other two parts are similar.

(a) $\ln e^4 = 4$.

(b) $\ln(e^{-3.27}) = -3.27$.

(c) $e = e^1$, so $\ln e = 1$.



Activity 24 *Exploring natural logarithms*

(a) Find the natural logarithms of the following numbers, without using your calculator.

(i) e^2 (ii) e^{-4} (iii) 1 (iv) $\dfrac{1}{e^2}$

(b) Given that $e^{5.37} = 214.86$ to two decimal places, write down the approximate value of $\ln 214.86$.

The next activity shows you how to use your calculator to find natural logarithms of numbers, and to find powers of e, which is the inverse operation.

Activity 25 *Natural logarithms and powers of e on your calculator*

Work through Subsection 3.11 of the MU123 Guide.

In the examples and activities in this section you have seen that

$$\log_{10} 1 = 0, \quad \log_2 1 = 0 \quad \text{and} \quad \ln 1 = 0.$$

In general, the logarithm of 1 to *any base* is 0. This is because if b is any base number, then $b^0 = 1$ and so the power to which b must be raised to give 1 is 0.

You have also seen that

$$\log_{10} 10 = 1, \quad \log_2 2 = 1 \quad \text{and} \quad \ln e = 1.$$

In general, for *any base* b, the logarithm of b to base b is 1. This is because $b^1 = b$ and so the power to which b must be raised to give b is 1.

These two facts are summarised below.

Logarithm of the number one and logarithm of the base

For any base b,

$$\log_b 1 = 0 \quad \text{and} \quad \log_b b = 1.$$

Benford's law

You might remember reading in Unit 3 that if you investigate the *first digits* of the numbers in a table of data, such as financial figures, then surprisingly the digits $1, 2, \ldots, 9$ do not occur equally often. Figure 30 shows the percentage occurrences of the nine digits.

It was mentioned in Unit 3 that there is a formula for the percentage occurrence of each digit, which is known as *Benford's law*. In fact this formula involves logarithms to base 10 and can be stated as

$$\begin{aligned}&\text{percentage occurrence of digit } N\\ &= (\log_{10}(N+1) - \log_{10} N) \times 100\%.\end{aligned}$$

For example, this formula gives the percentage occurrence of the digit 2 as

$$(\log_{10} 3 - \log_{10} 2) \times 100\% \approx 17.6\%,$$

and the percentage occurrence of the digit 7 as

$$(\log_{10} 8 - \log_{10} 7) \times 100\% \approx 5.8\%,$$

so 2 occurs about three times as often as 7 as a first digit.

There are many other instances of logarithms cropping up in real life where you might least expect them!

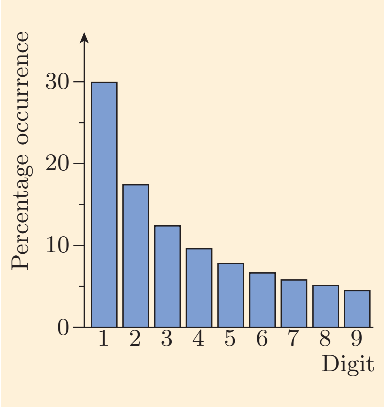


Figure 30 The percentage occurrences of $1, 2, \ldots, 9$ as the first digits of numbers in a table of data

In this section you have met the idea of logarithms. You have seen that taking a logarithm to base b is the inverse operation to raising the base b to a power. Any positive number except 1 can be used as the base of logarithms, but the two most commonly used bases are 10 and Euler's number e.

You have also learned about two measures that are based on logarithmic scales – the magnitude of an earthquake on the Richter scale, and the bel, which is a measure of sound intensity.

5 Working with logarithms

This section begins by introducing three rules that are useful when you are working with logarithms. Then you will learn how one of these rules can be used to help you to solve exponential equations, and you will find the answers to some problems by solving such equations. In the final subsection you will have a chance to explore the graphs of some functions of the form $y = \log_b x$.

5.1 Three logarithm laws

The three rules for logarithms that you will meet in this subsection depend on the following three index laws that you met in Unit 3.

Three index laws from Unit 3

$$a^m \times a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}.$$

You will be introduced to the three rules for logarithms one at a time.

Rule for addition of logarithms

The first rule for logarithms is illustrated in the calculation below, which you saw in Subsection 4.1. The calculation works because of the first index law in the pink box on the previous page.

$$\begin{aligned} 287 \times 37 &\approx 10^{2.4579} \times 10^{1.5682} \\ &= 10^{2.4579+1.5682} \\ &= 10^{4.0261} \\ &\approx 10\,619. \end{aligned}$$

This calculation shows that you can multiply 287 and 37 by first finding their logarithms to base 10, which are approximately 2.4579 and 1.5682, respectively, then adding these logarithms to obtain 4.0261, and finally finding the number whose logarithm to base 10 is 4.0261, which is approximately 10 619. In other words, as you can see by looking at the calculation, if you add the logarithms to base 10 of 287 and 37, then you obtain the logarithm to base 10 of their product, 287×37. Or, in the usual notation,

$$\log_{10} 287 + \log_{10} 37 = \log_{10}(287 \times 37).$$

You can see that a similar statement will hold for any two positive numbers – it is the reason why the logarithm method for multiplying numbers works. And the base of the logarithms does not have to be 10.

So in general the following law holds, where x and y are any positive numbers, and the logarithms can be to any base.

The base must be the same for each logarithm, of course.

Addition of logarithms

$$\log x + \log y = \log(xy)$$

You can use algebra to prove this result formally, as follows. Suppose that the numbers x and y can be written as b^m and b^n, respectively. That is,

$$x = b^m \quad \text{and} \quad y = b^n.$$

Then

$$m = \log_b x \quad \text{and} \quad n = \log_b y.$$

The product xy can be written as

$$xy = b^m \times b^n = b^{m+n}.$$

This tells you that the power to which b must be raised to give xy is $m + n$. That is,

$$\log_b(xy) = m + n = \log_b x + \log_b y.$$

which is the logarithm law above.

Rule for subtraction of logarithms

The second rule for logarithms is illustrated in the calculation at the top of the next page, which you also saw in Subsection 4.1. This calculation works because of the second index law in the pink box on page 157.

$$\begin{aligned}287 \div 37 &\approx 10^{2.4579} \div 10^{1.5682}\\ &= 10^{2.4579-1.5682}\\ &= 10^{0.8897}\\ &\approx 7.757.\end{aligned}$$

From this calculation you can see that if you subtract the logarithms to base 10 of 287 and 37, then you obtain the logarithm to base 10 of their quotient, $287 \div 37$. Or, in the usual notation,

$$\log_{10} 287 - \log_{10} 37 = \log_{10}\left(\frac{287}{37}\right).$$

Again, a similar equation will hold for any two positive numbers – it is the reason why the logarithm method for dividing numbers works, and again, the base of the logarithms does not have to be 10.

So in general the following law holds, where x and y are any positive numbers, and the logarithms can be to any base.

Subtraction of logarithms

$$\log x - \log y = \log\left(\frac{x}{y}\right)$$

You can formally prove this fact using an argument similar to that for the first logarithm law. You might like to try this in the next activity.

Activity 26 *Proving a logarithm law*

Prove that for any base b and any positive numbers x and y,

$$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right).$$

The next example shows you how you can apply the two logarithm laws that you have seen so far.

Example 8 *Combining logarithms*

Write each of the following expressions as a logarithm of a single number.

(a) $\log_{10} 4 + \log_{10} 5$ (b) $\ln 36 - \ln 9$ (c) $\ln 20 + \ln 3 - \ln 9$

Solution

Use the laws for addition and subtraction of logarithms.

(a) $\log_{10} 4 + \log_{10} 5 = \log_{10}(4 \times 5) = \log_{10} 20$

(b) $\ln 36 - \ln 9 = \ln\left(\frac{36}{9}\right) = \ln 4$

(c) $$\begin{aligned}\ln 20 + \ln 3 - \ln 9 &= \ln(20 \times 3) - \ln 9\\ &= \ln\left(\frac{20 \times 3}{9}\right)\\ &= \ln\left(\frac{20}{3}\right)\end{aligned}$$

Activity 27 *Combining logarithms*

Write each of the following expressions as a logarithm of a single number.

(a) $\log_{10} 3 + \log_{10} 4$ (b) $\ln 15 - \ln 3$

(c) $\log_{10} 2 + \log_{10} 3 + \log_{10} 5$ (d) $-\ln 4 + \ln 2$

Rule for a multiple of a logarithm

You have seen that, before the days of calculators and computers, logarithms made it easier to carry out multiplication and division calculations, because they allowed them to be turned into addition and subtraction calculations, respectively.

Logarithms could also be used to turn a calculation involving raising a number to a power into a multiplication calculation, as demonstrated by the calculation below. As with the earlier calculations, the initial step of finding the logarithm of 287 and the final step of finding the antilogarithm of 4.424 22 would have been carried out by using logarithm tables.

This calculation works because of the third index law in the pink box on page 157.

This third index law is $(a^m)^n = a^{mn}$.

$$\begin{aligned} 287^{1.8} &\approx \left(10^{2.4579}\right)^{1.8} \\ &= 10^{2.4579\times 1.8} \\ &= 10^{4.424\,22} \\ &\approx 26\,560. \end{aligned}$$

From this calculation you can see that if the logarithm to base 10 of 287 is multiplied by 1.8 then the result is the logarithm to base 10 of $287^{1.8}$. Or, in the usual notation,

$$1.8 \times \log_{10} 287 = \log_{10}\left(287^{1.8}\right).$$

A similar equation will hold for any positive number replacing 287, and any number at all replacing 1.8. And, as before, the base of the logarithms does not have to be 10.

So in general the following law holds, where x is any positive number, n is any number at all (including any negative number) and the logarithms can be to any base.

Multiple of a logarithm

$$n \log x = \log(x^n)$$

You can use algebra to prove this result formally, as follows. Suppose that the number x can be written as b^m. That is,

$$x = b^m.$$

Then

$$m = \log_b x.$$

The power x^n can be written as

$$x^n = (b^m)^n = b^{mn}.$$

This tells you that the power to which b must be raised to give x^n is mn. That is,

$$\log_b(x^n) = mn = n\log_b x,$$

which is the logarithm law above.

The example below shows you some ways in which you can use the third logarithm law.

Example 9 *Combining more logarithms*

Write each of the following expressions as a logarithm of a single number.

(a) $2\log_{10} 3$ (b) $\ln 24 - 3\ln 2$

Solution

Use the logarithm law on page 160. In part (b), also use the logarithm law on page 159.

(a) $2\log_{10} 3 = \log_{10}(3^2) = \log_{10} 9$

(b) $\ln 24 - 3\ln 2 = \ln 24 - \ln(2^3) = \ln\left(\dfrac{24}{2^3}\right) = \ln\left(\dfrac{24}{8}\right) = \ln 3$

Activity 28 *Combining more logarithms*

Write each of the following expressions as a logarithm of a single number.

(a) $3\log_{10} 3$ (b) $\ln 6 + 2\ln 5$ (c) $0.5\log_{10} 4$ (d) $\ln 27 - 2\ln 3$

Here is a summary of the three logarithm laws that you have seen in this section.

Three logarithm laws

$$\log x + \log y = \log(xy)$$

$$\log x - \log y = \log\left(\frac{x}{y}\right)$$

$$n\log x = \log(x^n)$$

As usual, these laws apply to all appropriate numbers. So n can be any number (in particular, it can be fractional and/or negative), but x and y must be positive, since only positive numbers have logarithms. These laws apply for logarithms to any base.

You might find the third of these three laws easier to remember if you notice how it is connected to the first law. The numbers x and y in the first law do not have to be different, of course. If they are the same, then you obtain

$$\log x + \log x = \log(x \times x),$$

that is,

$$2\log x = \log(x^2), \tag{4}$$

which is the third logarithm law with $n = 2$.

You can also add another $\log x$ to each side of equation (4), to give

$$2\log x + \log x = \log(x^2) + \log x.$$

Collecting the like terms on the left-hand side and using the first logarithm law again on the right-hand side gives

$$3\log x = \log(x^3),$$

which is the third logarithm law with $n = 3$.

You can see that this pattern will continue, to give the third logarithm law for any positive whole number n. But remember that the law holds not just if n is a positive whole number, but for *any* number n.

5.2 Solving exponential equations by taking logs

As you saw at the beginning of Section 4, an *exponential* equation is an equation in which the unknown is in an exponent, such as

$$170 \times 1.23^t = 7\,000\,000\,000.$$

You may remember that this particular equation arose in the context of finding the number of years that it would take for the entire population of the world to be Elvis impersonators.

In the absence of a better method, so far you have solved equations of this sort by using trial and improvement or graphs. You are now ready to learn how to use logarithms to solve exponential equations.

The method involves using the third logarithm law from the previous subsection. For this purpose it is best to think of the law with its left- and right-hand sides swapped:

$$\log(x^n) = n\log x.$$

The next example shows how you can use this logarithm law to help you to solve exponential equations.

Example 10 *Solving an exponential equation by taking logs*

Solve the equation $0.7^x = 0.2$, giving the solution to three significant figures.

Solution

The equation is

$$0.7^x = 0.2.$$

Take the logarithm to base 10 of both sides.

$$\log_{10}(0.7^x) = \log_{10} 0.2$$

Use the fact that $\log(x^n) = n\log x$.

$$x\log_{10} 0.7 = \log_{10} 0.2$$

Divide both sides by the coefficient of the unknown.

$$x = \frac{\log_{10} 0.2}{\log_{10} 0.7}$$

Use your calculator to evaluate the answer.

$$x = 4.51 \text{ (to 3 s.f.)}.$$

The solution is approximately $x = 4.51$.

(Check: When $x = 4.51$, LHS $= 0.7^{4.51} = 0.200\,16\ldots \approx 0.2 =$ RHS.)

So the basic procedure for solving an exponential equation is to take the logarithm of both sides of the equation – this is the 'taking logs' mentioned in the title of this subsection – and then apply the logarithm law mentioned on the opposite page to turn the awkward exponent into a straightforward coefficient. You can then solve the equation in the usual way.

You can use logarithms to any base in this procedure, as long as you are consistent. However, you will usually need to evaluate these logarithms, so it is best to use either logarithms to base 10 or natural logarithms, as these are easily available from your calculator.

Activity 29 *Solving an exponential equation by taking logs*

Solve the equation $1.4^x = 550$, giving the solution to three significant figures.

The method of taking logs is easiest to apply when one side of the equation consists only of a number raised to an exponent, where this exponent contains the unknown. So if you have to solve an exponential equation that is not in this form, then it is usually best to rearrange it into this form before you take logs.

This is demonstrated in the next example, which also illustrates the kind of problem that you can answer by solving exponential equations.

Example 11 *Solving another exponential equation*

Tutorial clip

A population of insects currently numbers 200. If the size of the population increases by 10% each week, how long will it take to reach 400?

Solution

The starting number is 200 and the scale factor is 1.1, so the size of the population is modelled by the equation

You saw how to obtain formulas like the one here in Section 1.

$$P = 200 \times 1.1^t,$$

where t is the number of weeks and P is the size of the population.

So the time t weeks for the population to reach 400 is given by the equation

$$200 \times 1.1^t = 400.$$

This equation can be solved as follows.

First divide both sides by 200 to obtain 1.1^t by itself on one side.

$$1.1^t = 2$$

Now take logs, and solve the equation using the method that you have seen.

$$\log_{10}(1.1^t) = \log_{10} 2$$
$$t \log_{10} 1.1 = \log_{10} 2$$
$$t = \frac{\log_{10} 2}{\log_{10} 1.1}$$
$$t = 7.27 \text{ (to 2 d.p.)}$$

So the insect population will take about $7\frac{1}{4}$ weeks to reach 400.

(Check: When $t = 7.27$, $P = 200 \times 1.1^{7.27} = 399.90\ldots \approx 400$.)

Before you go on to solve similar equations yourself, there is something worth observing about Example 11. If the starting population of insects had been 300, say, and the question had asked you to find the time taken for it to reach double this number, which is 600, then the answer would have been the same. This is because the answer would be found by solving the equation

$$300 \times 1.1^t = 600,$$

and this simplifies to

$$1.1^t = 2,$$

which is the same equation as in the example. In fact, you can see that whatever the starting population is, the time taken for it to double would be the same. So the time taken for the population to double does not depend on the starting population, but depends only on the scale factor. You will learn more about this observation in the next subsection.

You can practise solving exponential equations by taking logs in the next activity.

Activity 30 *Solving the Elvis impersonator problem*

Solve the equation

$$170 \times 1.23^t = 7\,000\,000\,000$$

by taking logs, giving your answer to three significant figures.

(The solution of this equation is the time in years that it would take for Elvis impersonators to account for the entire population of the world, assuming that their number starts at 170 and grows by 23% per year, and that the population of the world remains constant at 7 billion.)

The method of taking logs can also be used for problems involving *discrete* exponential change. For example, in Subsection 2.1 you looked at the case of an athlete who plans a new training schedule in which the distance in kilometres that she will run in week n is

$$20 \times 1.1^n.$$

The problem was to find the week of the schedule in which the athlete is first due to run more than 65 km. In other words, you have to find the smallest integer value of n for which

$$20 \times 1.1^n > 65.$$

This problem was solved by trial and improvement in Example 2 on page 132. A quicker way to solve it is to use the method of taking logs to find the solution of the equation

$$20 \times 1.1^n = 65;$$

then the required value of n is the smallest whole number greater than this solution. You might like to try this – you can check your answer against the answer found in Example 2.

In higher-level mathematics modules many exponential equations involve the base e.

If you have to solve an exponential equation in which the base is e, then it is usually helpful to take *natural logarithms* of both sides, rather than logarithms to base 10. This often allows you to obtain an exact answer in terms of natural logarithms, which you can then evaluate if necessary. This is illustrated in the next example.

In this example the exponent is not simply the unknown, x, but an expression involving the unknown, namely $2x$. Taking logs is useful in cases like this too, whatever the base is.

Example 12 *Solving an exponential equation in which the base is e*

Find the exact solution of the equation $e^{2x} = 3$, then evaluate it to three significant figures.

Solution

The equation is

$$e^{2x} = 3.$$

The base is e, so take natural logarithms.

$$\ln(e^{2x}) = \ln 3$$

Use the fact that taking the natural logarithm of a number is the inverse operation to raising e to a number.

$$2x = \ln 3$$

Now rearrange the equation in the usual way.

$$x = \tfrac{1}{2}\ln 3$$

Evaluate the answer if required.

$$x = 0.549 \text{ (to 3 s.f.)}$$

(Check: When $x = 0.549$, LHS $= e^{2\times 0.549} = 2.998\ldots \approx 3 =$ RHS.)

Another way to obtain the third equation in the solution here is to notice that the first equation tells you that you can obtain 3 by raising the base e to the power $2x$; in other words, the natural logarithm of 3 is $2x$.

Activity 31 *Solving exponential equations in which the base is e*

Find the exact solutions of the following equations, then evaluate them to three significant figures.

(a) $e^t = 8$ (b) $e^{x-1} = 5$

The useful fact used in Example 12 and Activity 31 was as follows. Suppose that you take a number x and first raise the base e to that number, then take the natural logarithm of the result. Since the two operations are inverses of each other, the final result will just be x, which gives the following identity:

Remember that an *identity* is an equation that is true for all appropriate values of its variable or variables.

$$\ln(e^x) = x.$$

You can also apply the operations in the other order. If you take a number x and first take the natural logarithm of that number, then raise the base e to the result, then again the final result will just be x, which gives the following second useful identity:

$$e^{\ln x} = x.$$

Similar facts to those in these two pink boxes hold for bases other than e.

5.3 Doubling and halving times

Example 11 in the previous subsection was about a population of insects that increased by 10% each week, and it was pointed out that the time that it would take for the population to double does not depend on the number of insects at the start.

In fact, not only is this true, but you could look at the number of insects *at any point in time after the starting time*, and the time that it would take for the population to double *from its value at that point* would be the same as the time taken for the population at the start to double.

This is a consequence of the property of continuous exponential change that you met in Subsection 2.2. You saw there that if a quantity changes by the scale factor b every week, for example, then over any time interval of length i weeks, the quantity changes by the scale factor b^i.

The population of insects changes by the scale factor 1.1 every week, so, for example, over any time interval of length 3 weeks, it changes by the scale factor

$$1.1^3 \approx 1.33,$$

and similarly over any time interval of length 10 weeks, say, it changes by the scale factor

$$1.1^{10} \approx 2.59,$$

and so on. You can see that there must be a particular length of time in which the population changes by the scale factor 2 – that is, it *doubles* – and that this length of time, in weeks, will be the number i that satisfies the equation

$$1.1^i \approx 2.$$

This equation is the one that was solved in Example 11. It was solved by taking logs, and its solution was found to be about 7.27. So whenever you record the size of the population of insects, the population will have doubled about 7.27 weeks later.

You can see that you could apply the same argument to any example involving continuous exponential growth. So, for any quantity that is subject to continuous exponential growth over time, there will be a particular length of time in which its size always doubles. This length of time is called its **doubling time**.

The doubling time can be worked out in a similar way to the insect example. For example, if the quantity increases by the scale factor 1.8 each week, then the doubling time, in weeks, is the number i that satisfies the equation $1.8^i = 2$. Similarly, if the quantity increases by the scale factor 1.05 each year, then the doubling time, in years, is the number i that satisfies the equation $1.05^i = 2$. As you know, you can solve equations like these by taking logs.

Activity 32 *Finding a doubling time*

If the number of Elvis impersonators is increasing by 23% per year, what is its doubling time? Give your answer in years to two significant figures.

You might have heard of *Moore's law*, which suggests that the number of transistors on a typical microprocessor (computer chip) increases exponentially with time, and doubles approximately every two years. The number of transistors on a microprocessor is an important quantity, because it strongly influences the performance of the electronic device containing the microprocessor – for example, it affects the processing speed of a computer. Microprocessor manufacturers have largely kept to the rate of increase in Moore's law, or faster, since Gordon Moore first suggested this law in 1965, at least up until the time of writing of MU123. However, it will probably not be sustainable for much longer, because of physical limits in manufacturing processes.

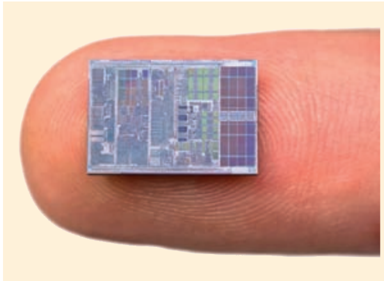


Figure 31 A microprocessor

A quantity that is subject to continuous exponential decay, rather than growth, will not have a doubling time, of course, but it will have a **halving time**.

For example, consider again the car discussed on page 135, whose value immediately after purchase is £20 000 and which depreciates at a rate of 20% per year. The value of the car depreciates by the scale factor 0.8 each year, so, for example, every two years it depreciates by the scale factor

$$0.8^2 = 0.64,$$

and every 3.5 years it depreciates by the scale factor

$$0.8^{3.5} \approx 0.46,$$

and so on. There will be a particular length of time in which the value of the car depreciates by the scale factor $\frac{1}{2}$ – that is, it halves – and this length of time, in years, will be the number i that satisfies the equation

$$0.8^i = \tfrac{1}{2}.$$

In other words, this number i is the halving time, in years, of the value of the car. The equation can be solved by taking logs, as follows.

$$\log_{10}\left(0.8^i\right) = \log_{10} 0.5$$
$$i \log_{10} 0.8 = \log_{10} 0.5$$
$$i = \frac{\log_{10} 0.5}{\log_{10} 0.8} \approx 3.1$$

So the value of the car halves every 3.1 years, approximately.

Activity 33 *Finding a halving time*

A population of elephants is thought to be declining at the rate of 4% per year. Find the scale factor by which the population decreases every year, and hence find its halving time, to the nearest year.

The halving time of a quantity that decays exponentially is often called its **half-life**. Both doubling times and halving times (half-lives) are useful as descriptions of the speed of exponential growth or decay. For example, doctors can use information about the typical half-lives of prescription drugs in patients' bloodstreams to help them to make decisions about the frequency of doses.

The ideas about doubling time and halving time that you have met in this subsection are summarised below.

Doubling time and halving time (half-life)

Suppose that a quantity is subject to continuous exponential change by the scale factor b every year.

If the quantity grows (that is, if $b > 1$), then its doubling time in years is the solution i of the equation

$$b^i = 2.$$

If the quantity decays (that is, if $0 < b < 1$), then its halving time (half-life) in years is the solution i of the equation

$$b^i = \tfrac{1}{2}.$$

(If the quantity changes by the scale factor b every *week*, then the solution i of the equation is the halving time in *weeks*, and similarly for any other units of time, such as days or minutes.)

The half-lives of radioactive isotopes range from tiny fractions of a second to billions of years. Knowledge of the half-lives of some radioactive isotopes can be used to find out how old objects are. For example, the isotope carbon-14 has a half-life of 5730 years, and measurements of carbon-14 are used in carbon dating. This process can determine the approximate age of organic material, such as plant and animal remains, up to several tens of thousands of years old.

You may have heard the term half-life used in connection with radioactivity. Many chemical elements have several different forms, called *isotopes*, which differ in the number of a particular type of subatomic particle, called a *neutron*, in their atoms. Some chemical isotopes are radioactive, which means that over time they decay atom by atom into isotopes of a different element, emitting radiation as they do so.

Scientists have found that when a sample of a radioactive isotope decays, the amount of the original radioactive isotope remaining can be modelled by an exponential decay curve. The speed of the decay is usually described by giving the half-life. So, for example, if you are told that the half-life of a radioactive isotope is 100 years, then this means that the amount of the radioactive isotope halves every 100 years.

If a radioactive isotope decays into a non-radioactive isotope of another element, then the level of radioactivity drops in proportion to the amount of the radioactive isotope, so the level of radioactivity can also be modelled by an exponential decay curve, with the same halving time. This is the topic of the next activity.

Activity 34 *Finding a half-life*

A *becquerel* is one radioactive disintegration per second. This unit of measurement is named after the French physicist Henri Becquerel (1852–1908), who discovered radioactivity in 1896. He was awarded the 1903 Nobel prize for physics, along with Marie Curie and Pierre Curie, who found additional radioactive elements.

The level of radioactivity of a piece of material containing a single radioactive isotope was initially 40 becquerels. One day later the level had fallen to 38 becquerels. Assume that the radioactive isotope decays into a non-radioactive substance.

(a) Find the scale factor by which the radioactivity decreases each day.

(b) Hence find the half-life of the radioactive isotope, in days to one decimal place.

5.4 Graphs of logarithmic functions

As you know, the operations of raising a base b to a number and taking the logarithm to the same base b of a number are inverse operations. Another

way to express this is to say that the functions with rules

$$y = b^x \quad \text{and} \quad y = \log_b x \qquad (5)$$

are *inverse functions*. In general, **inverse functions** are functions whose rules undo the effects of each other.

In the final activity of this section you are asked to use Graphplotter to plot the graphs of some pairs of inverse functions of form (5). You should be able to gain a few useful insights, not just about these particular graphs but also about the connection between the graphs of any two inverse functions.

A function with a rule of the form $y = \log_b x$ is called a **logarithmic function**.

Activity 35 *Investigating graphs of equations of the form $y = \log_b x$ and $y = b^x$*

Graphplotter

Use Graphplotter, with the 'Two graphs' tab selected. Click the 'Autoscale' button to ensure that the axis scales are at their default values. Make sure that the 'y-intercept' option is switched off – it is not needed in this activity.

(a) Choose the equation

$$y = ab^x + c$$

from the left-hand drop-down list, and make sure that $a = 1$ and $c = 0$, since this activity is about the graphs of equations of the form $y = b^x$. Then choose the equation

$$y = a\log_b(cx) + d$$

from the right-hand drop-down list, and make sure that $a = 1$, $c = 1$ and $d = 0$, since this activity is also about the graphs of equations of the form $y = \log_b x$.

Keep all these constants set to these values throughout the activity. You will need to change only the value of the base b for each graph.

(b) Set $b = 10$ for both equations to obtain the graphs of the equations

$$y = 10^x \quad \text{and} \quad y = \log_{10} x.$$

Make a note of how the two graphs seem to be related.

(c) Repeat part (b) for the equations

$$y = 2^x \quad \text{and} \quad y = \log_2 x.$$

(d) Repeat part (b) for the equations

$$y = e^x \quad \text{and} \quad y = \ln x.$$

You will see that there are checkboxes to set $b = e$.

(e) Repeat part (b) for the equations

$$y = 0.5^x \quad \text{and} \quad y = \log_{0.5} x.$$

In each part of Activity 35, you should have found that the two graphs are mirror images of each other, reflected in the line $y = x$. For example, Figure 32 (overleaf) shows the graphs of $y = 2^x$ and $y = \log_2 x$, with the line $y = x$ shown in red dashes.

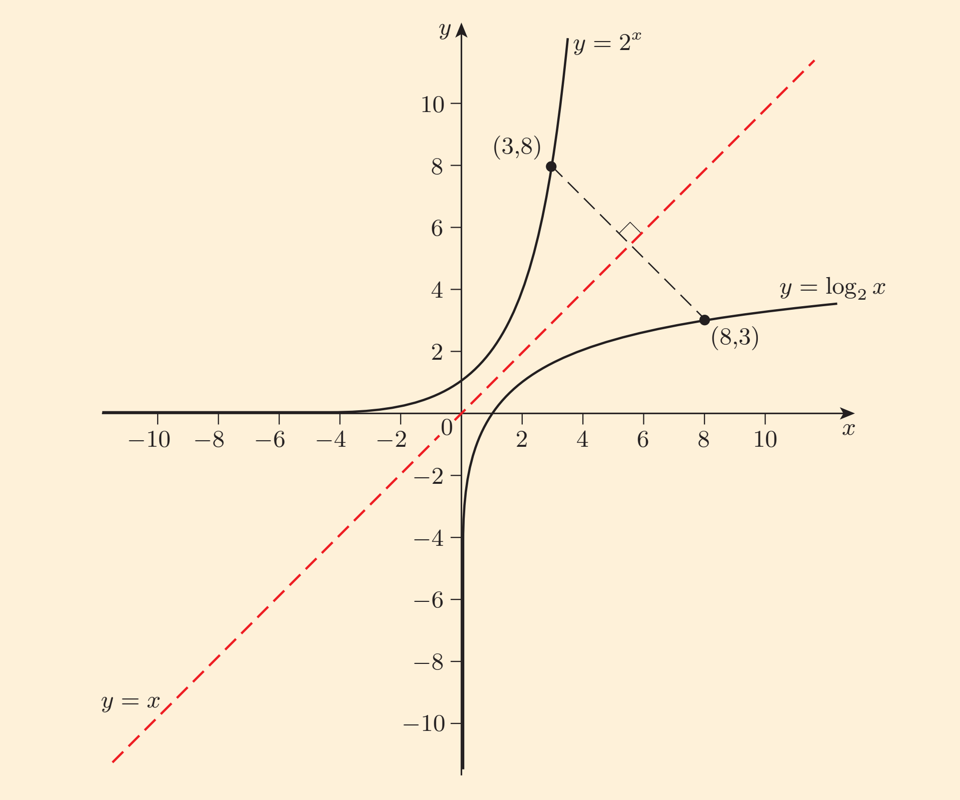


Figure 32 The graphs of the equations $y = 2^x$ and $y = \log_2 x$ are reflections of each other in the line $y = x$

To see why this symmetry occurs, consider a particular point on the graph of $y = 2^x$. For example, the point $(3, 8)$ lies on this graph, because inputting 3 to the rule $y = 2^x$ gives the output 8. Since the function with rule $y = \log_2 x$ is the inverse of the function with rule $y = 2^x$, inputting 8 to *this* function gives the output 3, and so the point $(8, 3)$, obtained by swapping the coordinates of the first point, lies on *its* graph. The two points are the reflections of each other in the line $y = x$, as shown in Figure 32.

In general, if you swap the coordinates of any point on the graph of $y = 2^x$, then you obtain the coordinates of a point on the graph of $y = \log_2 x$, and vice versa. And when you swap the coordinates of a point, the resulting point is always the reflection of the first point in the line $y = x$. This explains why the two graphs are reflections of each other in this line.

You can see that the same argument will apply to any pair of functions of the form $y = b^x$ and $y = \log_b x$, for any base b, and indeed it will apply to any pair of inverse functions. So we have the following fact.

Graphs of inverse functions

The graphs of a pair of inverse functions are reflections of each other in the line $y = x$.

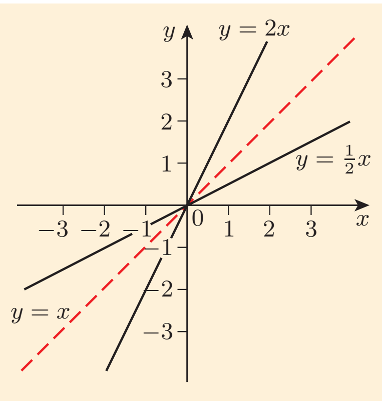


Figure 33 The lines $y = 2x$ and $y = \frac{1}{2}x$ are reflections of each other in the line $y = x$

For example, Figure 33 shows the graphs of the equations $y = 2x$ and $y = \frac{1}{2}x$. These follow the rules of inverse functions, since halving a number reverses the effect of doubling it, and vice versa. The line $y = x$ has been drawn in red dashes, and you can see that each of the graphs is a reflection of the other in this line.

If you know some features of the graph of a particular function, then you can use the property in the pink box on the opposite page to deduce some corresponding features of the graph of its inverse function. In particular, knowing what the graph of an exponential function looks like allows you to deduce some properties of the graph of the logarithmic function that is its inverse.

For example, you know that the graph of an equation of the form $y = b^x$ has the x-axis as an asymptote, unless $b = 1$. It follows that the graph of an equation of the form $y = \log_b x$ has the y-axis as an asymptote. Figure 32 illustrates this for the base $b = 2$.

Similarly, you know that the graph of $y = b^x$ has y-intercept 1. It follows that the graph of $y = \log_b x$ has x-intercept 1. Again, this is illustrated for $b = 2$ in Figure 32.

Also, you know that if $b > 1$, then the graph of $y = b^x$ increases, and increases more and more quickly. If you think about it (drawing a diagram helps), then you will see that this means that if $b > 1$, then the graph of $y = \log_b x$ increases, but increases more and more slowly. Again, you can see this illustrated for the base $b = 2$ in Figure 32.

In this section you have met some rules that you can use when you are working with logarithms. You have also learned how to solve exponential equations by taking logs. In the final subsection you saw that, for any base b, the graphs of the equations $y = b^x$ and $y = \log_b x$ are reflections of each other in the line $y = x$, and that this property applies to all pairs of inverse functions.

Learning checklist

After studying this unit, you should be able to:

- understand what is meant by exponential change
- recognise some standard examples of exponential growth and decay, such as repeated doubling or halving, compound interest and radioactive decay
- find and use formulas for exponential growth and decay
- use trial and improvement to solve problems
- understand the relationship between the scale factor by which a quantity changes over a particular length of time (or over one step) and the scale factor by which it changes over a different length of time (or different number of steps)
- know and understand the shapes of the graphs of equations of the form $y = b^x$ and $y = ab^x$
- understand logarithms to any base (in particular to base 10 and base e) and use appropriate notation
- use logarithm laws to manipulate expressions involving logarithms
- solve exponential equations by taking logs
- calculate doubling or halving times for quantities that are subject to continuous exponential growth or decay
- understand that exponential and logarithmic functions form pairs of inverse functions.

Solutions and comments on Activities

Activity 1

You might be surprised by the true answer for the height of the pile of paper. You can work it out, roughly, as follows. The thickness of a piece of paper is about one tenth of a millimetre, or about 0.0001 metres, so the height of the initial pile of one sheet of paper is 0.0001 metres. After each step of the process, the height of the paper pile doubles. So the heights, in metres, of the successive piles of paper are as follows:

the initial height is 0.0001;

after 1 step, the height is

$0.0001 \times 2;$

after 2 steps, the height is

$0.0001 \times 2 \times 2 = 0.0001 \times 2^2;$

after 3 steps, the height is

$0.0001 \times 2 \times 2 \times 2 = 0.0001 \times 2^3;$

and so on. In general, after n steps the height in metres of the paper pile is

$$0.0001 \times 2^n.$$

So after 50 steps the height is

$$0.0001 \times 2^{50}\,\text{m}.$$

The number 2^{50}, which is the number of pieces of paper, is approximately 1.13×10^{15} in scientific notation, or 1 130 000 000 000 000 in ordinary notation. So the height of the pile is roughly

$$\begin{aligned} 0.0001 \times 1.13 \times 10^{15}\,\text{m} &= 10^{-4} \times 1.13 \times 10^{15}\,\text{m} \\ &= 1.13 \times 10^{11}\,\text{m} \\ &= 1.13 \times 10^{8}\,\text{km}, \end{aligned}$$

which is 113 million kilometres.

The average distance to the moon is about 384 000 km. So, if you could achieve the number of tears of a single piece of paper suggested in this activity, which in practice would, of course, be impossible, then the resulting pile of paper would correspond to roughly 150 trips to the moon and back!

Activity 2

The correct answer is that the pond was half covered after 29 days. If you are unsure why, consider the following statement.

> If the area covered by the lily pads was twice as large on Thursday as it was on Wednesday, then it must have been half as large on Wednesday as it was on Thursday.

This demonstrates why the area covered by the lily pads was half of the size of the pond on the penultimate day (day 29): one day later, the area covered had doubled to fill the pond.

(A common wrong answer to this question is that it would take 15 days for the lily pads to cover half of the pond. This would be the correct answer if the area covered by the lily pads were increasing linearly – that is, by a fixed amount each day – and if the lily pads did not cover any of the pond at the start of the 30 days.)

Activity 3

(a) The stallholder's selling price is 165% of his buying price. So he must multiply the buying price by the scale factor $\frac{165}{100} = 1.65$.

So the selling price for a purse is

$$£4.25 \times 1.65 \approx £7.01.$$

(In practice, the stallholder may round the price to £7 or even to £6.99.)

(b) The stallholder's 30% discount means that he is selling at 70% of his usual price. So he must multiply the usual price by the scale factor $\frac{70}{100} = 0.7$.

The discounted price for a shirt is

$$£7 \times 0.7 = £4.90.$$

Activity 4

(a) **(i)** The scale factor for a 10% increase is 1.1.

(ii) The scale factor for a 3% increase is 1.03.

(iii) The scale factor for a 0.5% increase is 1.005.

(iv) The scale factor for a 15% decrease is 0.85.

(v) The scale factor for a 2% decrease is 0.98.

(vi) The scale factor for a 1.5% decrease is 0.985.

(b) **(i)** A scale factor of 1.08 gives a 8% increase.

(ii) A scale factor of 0.91 gives a 9% decrease.

(iii) A scale factor of 1.072 gives a 7.2% increase.

Activity 5

(a) **(i)** After 1 year, the investment will be worth

$$£1800 \times 1.045 = £1881.$$

(ii) After 3 years, the investment will be worth

$$£1800 \times 1.045 \times 1.045 \times 1.045 = £2054.10$$

(to the nearest penny).

(iii) After 10 years, the investment will be worth

$$£1800 \times 1.045^{10} = £2795.34$$

(to the nearest penny).

(b) **(i)** After 1 year, the interest earned is

$$£1881 - £1800 = £81.$$

(ii) After 3 years, the interest earned is

$£2054.10 - £1800 = £254.10$

(to the nearest penny).

(iii) After 10 years, the interest earned is

$£2795.34 - £1800 = £995.34$

(to the nearest penny).

(c) A formula for the value $£V$ of the investment after n years is

$$V = 1800 \times 1.045^n.$$

A formula for the amount $£W$ of interest earned after n years is

$$W = 1800 \times 1.045^n - 1800.$$

This formula can be simplified slightly, by taking out the common factor 1800:

$$W = 1800(1.045^n - 1).$$

Activity 6

(a) The value, in £, of the investment in 18 years' time will be

$$M \times 1.05^{18}.$$

(b) Hence M must satisfy the equation

$$M \times 1.05^{18} = 1000.$$

This gives

$$M = \frac{1000}{1.05^{18}}.$$

Evaluating this expression on a calculator gives

$$M = 415.52 \text{ (to 2 d.p.)}.$$

Hence you must invest £415.52.

Activity 7

(a) The concentration decreases by 14% each hour. Since $100\% - 14\% = 86\%$, it decreases by the scale factor 0.86 each hour.

Also, the peak concentration is 90 ng/ml, so the concentration of the drug in the patient's bloodstream is given by the formula

$$C = 90 \times 0.86^t,$$

where t is the time in hours since the concentration peaked, and C is the concentration in ng/ml.

(b) **(i)** The concentration after 4 hours is

$$90 \times 0.86^4 \text{ ng/ml}$$
$$= 49.230\ldots \text{ ng/ml}$$
$$= 50 \text{ ng/ml (to the nearest 5 ng/ml)}.$$

(ii) The concentration after 30 minutes (0.5 hours) is

$$90 \times 0.86^{0.5} \text{ ng/ml}$$
$$= 83.462\ldots \text{ ng/ml}$$
$$= 85 \text{ ng/ml (to the nearest 5 ng/ml)}.$$

(c) The peak concentration occurs 40 minutes after the drug was administered. This means that two hours after the drug was administered is the same as 1 hour and 20 minutes after the concentration peaked. Now 1 hour and 20 minutes is equal to $1\frac{1}{3}$ hours; that is, $\frac{4}{3}$ hours. So the concentration at this time is

$$90 \times 0.86^{4/3} \text{ ng/ml}$$
$$= 73.604\ldots \text{ ng/ml}$$
$$= 75 \text{ ng/ml (to the nearest 5 ng/ml)}.$$

Activity 8

(a) The peak concentration is 36 μg/ml and the concentration after one hour is 27 μg/ml, so the scale factor is

$$\frac{27}{36} = 0.75.$$

(b) Using the answer to part (a), together with the value of the peak concentration, gives the formula

$$C = 36 \times 0.75^t.$$

(c) Since 1 hour and 15 minutes is the same as 1.25 hours, the expected concentration after this time is

$$36 \times 0.75^{1.25}\ \mu\text{g/ml}$$
$$= 25.126\ldots\ \mu\text{g/ml}$$
$$= 25\ \mu\text{g/ml (to the nearest } \mu\text{g/ml)}.$$

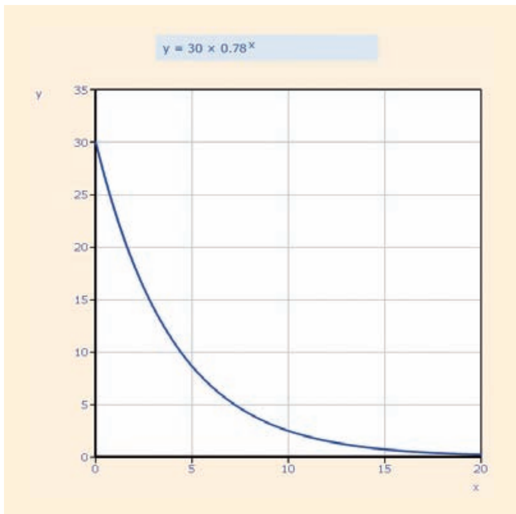
Activity 9

(a) The three most recent data points (corresponding to the years 1990, 2000 and 2009) all lie above the regression curve.

(b) Possible explanations might include improved preventative medicine, medical treatments and nutrition, leading to reduced death rates.

Activity 10

(b) The Graphplotter graph is shown below.



(c) By using the Trace facility you can find that the point on the graph with x-coordinate 4.4 has y-coordinate greater than 10, and the point on the graph with x-coordinate 4.5 has y-coordinate less than 10. Since both of these x-coordinates are 4.5 to the nearest 0.5, the point on the graph with y-coordinate 10 also has x-coordinate 4.5 to the nearest 0.5.

Hence the concentration falls to 10 μg/ml after about $4\frac{1}{2}$ hours.

Activity 11

(a) The scale factor by which the number of Elvis impersonators is increasing each year is 1.23. Hence the number of Elvis impersonators t years after 16 August 1977 is given by the formula

$$170 \times 1.23^t.$$

(b) By the formula in part (a), the number of Elvis impersonators 30 years after 16 August 1977 is

$$170 \times 1.23^{30} = 85000 \text{ (to the nearest 1000).}$$

(c) Your trial-and-improvement process might have been similar to that shown in the table below.

Guess for number of years	Number of Elvis impersonators	Evaluation
40	170×1.23^{40} $= 670\,893$	Much too small
80	170×1.23^{80} $= 2\,647\,633\,311$	Much closer but still too small
85	170×1.23^{85} $= 7\,453\,897\,111$	Just too big
84	170×1.23^{84} $= 6\,060\,078\,952$	Just too small

This calculation suggests that the entire population of the world will be Elvis impersonators roughly 84 to 85 years after his death. Since he died in 1977, the expected calendar year for this to happen is about 2061 or 2062.

Activity 12

(a) (i) The APR for an interest rate of 17.5%, charged annually, is 17.5%.

(ii) An interest rate of 8.5% each six months corresponds to a scale factor of 1.085 each six months. This gives a scale factor of

$$1.085^2 = 1.177 \text{ (to 3 d.p.)}$$

per year, which gives an interest rate of 17.7% per year. That is, the APR is approximately 17.7%.

(iii) An interest rate of 1.4% each month corresponds to a scale factor of 1.014 each month. This gives a scale factor of

$$1.014^{12} = 1.182 \text{ (to 3 d.p.)}$$

per year, which gives an interest rate of 18.2% per year. That is, the APR is approximately 18.2%.

(b) The interest rate in part (a)(i) would result in you paying the least interest.

Activity 13

The car depreciates at the rate of 16% per year, so its values change by the scale factor 0.84 each year.

(a) The value of the car one year after purchase was £15 450, so 0.25 years later its value was

$$£15\,450 \times 0.84^{0.25} = £14\,790 \text{ (to the nearest £10).}$$

(b) The value of the car 2.75 years after purchase, which is 1.75 years after it had a value of £15 450, was

$$£15\,450 \times 0.84^{1.75} = £11\,390 \text{ (to the nearest £10).}$$

(c) The value of the car at purchase was

$$£15\,450 \times 0.84^{-1} = £18\,390 \text{ (to the nearest £10).}$$

Activity 14

(a) Each decade, the number of pigeons grows by a scale factor of 1.8.

(b) Each year, the number of pigeons grows by a scale factor of

$$1.8^{1/10} = 1.0605\ldots = 1.061 \text{ (to 4 s.f.).}$$

(c) Each year, the number of pigeons grows by 6.1% (to 1 d.p.).

Activity 15

(a) The function with rule $y = 2x$ is linear, because this equation is of the form $y = mx + c$, with $m = 2$ and $c = 0$.

The function with rule $y = x^2$ is quadratic, because this equation is of the form $y = ax^2 + bx + c$, with $a = 1$ and $b = c = 0$.

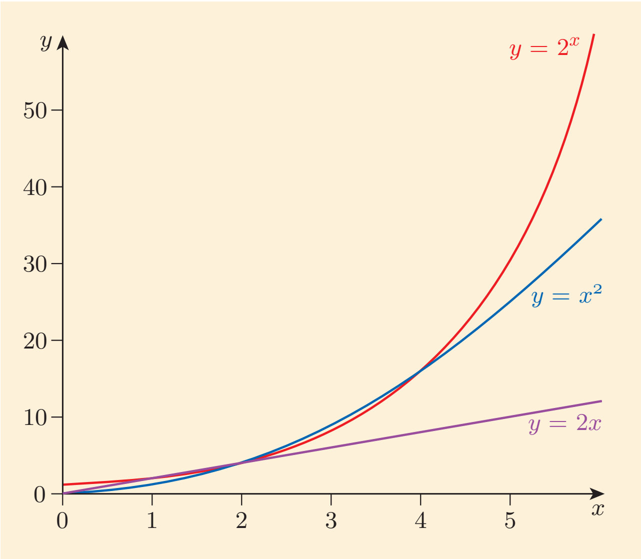
The function with rule $y = 2^x$ is exponential, because this equation is of the form $y = b^x$, with $b = 2$.

(b) Here is the completed table.

x	0	1	2	3	4	5	6
$y = 2x$	0	2	4	6	8	10	12
$y = x^2$	0	1	4	9	16	25	36
$y = 2^x$	1	2	4	8	16	32	64

The largest increase is that of the exponential function $y = 2^x$. The smallest increase is that of the linear function $y = 2x$.

(You can get a better idea of the growth of the three functions by looking at their graphs plotted on the same axes, as shown below.



You can see that in the range $0 < x < 2$, there is not much difference in the growth rates of the three functions. In the range $2 < x < 4$, the linear function $y = 2x$ starts to lag behind the other two, and it continues to do so thereafter. For $x > 4$, the quadratic function $y = x^2$ starts to lag behind the exponential function $y = 2^x$.

For larger values of x, the differences become really dramatic. For example, when $x = 20$, the value of $2x$ is 40, the value of x^2 is 400, and the value of 2^x exceeds one million. Increase x to, say, 100 and the value of the exponential function $y = 2^x$ disappears into the stratosphere!)

Activity 16

(c) As expected, if $b > 1$ then the graph of $y = b^x$ is an exponential growth curve, while if $0 < b < 1$ then the graph is an exponential decay curve.

Setting b to exactly 1 gives a horizontal straight line with y-intercept 1. The reason for this is that if $b = 1$ then the equation $y = b^x$ becomes $y = 1^x$, which is the same as $y = 1$. This is not an exponential function.

Varying the value of b affects the steepness of the graph. In each of the ranges $0 < b < 1$ and $b > 1$, the closer the value of b is to 1, the flatter is the graph.

(d) The y-intercept is always 1, whatever the value of b.

This is because substituting $x = 0$ into the equation $y = b^x$ always gives $y = b^0 = 1$, no matter what the value of b is.

Activity 17

(b) As expected, since $b = 2$ (which is greater than 1), if a is positive then the shape of the graph is always an exponential growth curve. Varying the value of a affects the steepness of the graph: the larger the value of a, the steeper the graph.

If a is negative, then the shape of the graph is the reflection in the x-axis of an exponential growth curve. Varying a within negative values affects the steepness of the graph: the larger the magnitude of a, the steeper the graph.

Setting $a = 0$ gives the horizontal line $y = 0$.

Whether a is positive, negative or zero, the y-intercept is always a.

(The reason for this is given after the activity.)

(c) If a is positive, then, as expected, the shape of the graph is always an exponential growth curve, exponential decay curve or horizontal line, depending on whether $b > 1$, $0 < b < 1$ or $b = 1$.

As in part (b), varying a within positive values affects the steepness of the graph: the larger the value of a, the steeper the graph.

If a is negative, then the shape of the graph is the reflection in the x-axis of an exponential growth curve, exponential decay curve or horizontal line, depending on whether $b > 1$, $0 < b < 1$ or $b = 1$.

Varying a within negative values affects the steepness of the graph: the larger the magnitude of a, the steeper the graph.

Setting $a = 0$ gives the horizontal line $y = 0$.

In all cases, whether a is positive, negative or zero, and whatever the value of b, the y-intercept is a.

Activity 18

You should have chosen a positive value for a and a value between 0 and 1 (exclusive) for b. For example, you could have chosen $a = 3$ and $b = 0.5$, which gives the rule

$$y = 3 \times 0.5^x.$$

The graph of this equation is shown in Figure 22(b) on page 144.

(A common error is to assume that negative values for the constant a give exponential decay.)

Activity 19

(c) Setting b to about 2.7 seems to give a gradient of 1 at $(0, 1)$.

Activity 20

(a) **(i)** $10 = 10^1$, so $\log_{10} 10 = 1$.

(ii) $1000 = 10^3$, so $\log_{10} 1000 = 3$.

(iii) 1 billion $= 10^9$, so $\log_{10}$ (1 billion) $= 9$.

(iv) $0.001 = 10^{-3}$, so $\log_{10} 0.001 = -3$.

(b) It is not possible to find the logarithm to base 10 of zero. The reason is that there is no power to which 10 can be raised to give the answer zero.

(If you try to find the logarithm of zero on your calculator, you will get an error message.)

(c) **(i)** The number 91 lies between 10 and 100, that is, between 10^1 and 10^2. So $\log_{10} 91$ lies between 1 and 2.

(ii) The number 2971 lies between 1000 and 10 000, that is, between 10^3 and 10^4. So $\log_{10} 2971$ lies between 3 and 4.

(iii) The number 8.8 lies between 1 and 10, that is, between 10^0 and 10^1. So $\log_{10} 8.8$ lies between 0 and 1.

(iv) The number 0.25 lies between 0.1 and 1, that is, between 10^{-1} and 10^0. So $\log_{10} 0.25$ lies between -1 and 0.

(d) If the logarithm to base 10 of a number is 5, then the number is $10^5 = 100\,000$.

Similarly, if the logarithm to base 10 of a number is -2, then the number is $10^{-2} = 0.01$.

Activity 22

(a) The two shocks differ by 2 on the Richter scale, and $10^2 = 100$, so the energy released by the aftershock was one hundredth of the energy released by the original shock.

(b) The noise levels of 6 bels and 7 bels differ by 1, so the sound intensity increases by a factor of $10^1 = 10$.

Activity 23

(a) $8 = 2^3$, so $\log_2 8 = 3$.

(b) $32 = 2^5$, so $\log_2 32 = 5$.

(c) $2 = 2^1$, so $\log_2 2 = 1$.

(d) $1 = 2^0$, so $\log_2 1 = 0$.

(e) $0.5 = \frac{1}{2} = 2^{-1}$, so $\log_2 0.5 = -1$.

(f) $\sqrt{2} = 2^{0.5}$, so $\log_2 \sqrt{2} = 0.5$.

Activity 24

(a) **(i)** $\ln(e^2) = 2$

(ii) $\ln(e^{-4}) = -4$

(iii) $\ln 1 = \ln(e^0) = 0$

(iv) $\ln\left(\dfrac{1}{e^2}\right) = \ln(e^{-2}) = -2$

(b) $\ln 214.86 \approx \ln(e^{5.37}) = 5.37$

Activity 26

Suppose that the numbers x and y can be written as b^m and b^n, respectively. That is,

$$x = b^m \quad \text{and} \quad y = b^n.$$

Then

$$m = \log_b x \quad \text{and} \quad n = \log_b y.$$

The quotient x/y can be written as

$$\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n}.$$

This tells you that

$$\log_b\left(\frac{x}{y}\right) = m - n = \log_b x - \log_b y.$$

Activity 27

(a) $\log_{10} 3 + \log_{10} 4 = \log_{10}(3 \times 4) = \log_{10} 12$

(b) $\ln 15 - \ln 3 = \ln\left(\dfrac{15}{3}\right) = \ln 5$

(c)
$$\begin{aligned}\log_{10} 2 + \log_{10} 3 + \log_{10} 5 &= \log_{10}(2 \times 3 \times 5)\\ &= \log_{10} 30\end{aligned}$$

(d) $-\ln 4 + \ln 2 = \ln 2 - \ln 4 = \ln\left(\dfrac{2}{4}\right) = \ln 0.5$

Activity 28

(a) $3\log_{10} 3 = \log_{10}(3^3) = \log_{10} 27$

(b)
$$\begin{aligned}\ln 6 + 2\ln 5 &= \ln 6 + \ln(5^2)\\ &= \ln(6 \times 5^2)\\ &= \ln 150\end{aligned}$$

(c)
$$\begin{aligned}0.5\log_{10} 4 &= \log_{10}\left(4^{0.5}\right)\\ &= \log_{10}(\sqrt{4})\\ &= \log_{10} 2\end{aligned}$$

(d)
$$\begin{aligned}\ln 27 - 2\ln 3 &= \ln 27 - \ln(3^2)\\ &= \ln\left(\frac{27}{3^2}\right)\\ &= \ln\left(\frac{27}{9}\right)\\ &= \ln 3\end{aligned}$$

Activity 29

The equation is

$$1.4^x = 550.$$

Taking logs gives

$$\log_{10}(1.4^x) = \log_{10} 550$$
$$x \log_{10} 1.4 = \log_{10} 550$$
$$x = \frac{\log_{10} 550}{\log_{10} 1.4}$$
$$x = 18.753\ldots.$$

So the solution is $x = 18.8$ (to 3 s.f.).

(Check: When $x = 18.8$,

$\text{LHS} = 1.4^{18.8} = 558.73\ldots \approx 550 = \text{RHS}$.)

Activity 30

The equation is

$$170 \times 1.23^t = 7\,000\,000\,000.$$

Dividing by 170 gives

$$1.23^t = \frac{7\,000\,000\,000}{170}.$$

Taking logs gives

$$\log_{10}(1.23^t) = \log_{10}\left(\frac{7\,000\,000\,000}{170}\right)$$
$$t \log_{10} 1.23 = \log_{10}\left(\frac{7\,000\,000\,000}{170}\right)$$
$$t = \log_{10}\left(\frac{7\,000\,000\,000}{170}\right) \div \log_{10} 1.23$$
$$t = 84.696\ldots.$$

So the solution is $t = 84.7$ (to 3 s.f.).

(Notice that this is consistent with the solution to Activity 11. That is, it will take 84 to 85 years for Elvis impersonators to account for the entire population of the world, under the assumptions stated in the question.)

Activity 31

(a) $e^t = 8$

$$\ln\left(e^t\right) = \ln 8$$
$$t = \ln 8$$

The solution is $t = 2.08$ (to 3 s.f.).

(Check: When $t = 2.08$,
$\text{LHS} = e^{2.08} = 8.004\ldots \approx 8 = \text{RHS}$.)

(b) $e^{x-1} = 5$

$$\ln\left(e^{x-1}\right) = \ln 5$$
$$x - 1 = \ln 5$$
$$x = \ln 5 + 1$$

The solution is $x = 2.61$ (to 3 s.f.).

(Check: When $x = 2.61$,
$\text{LHS} = e^{2.61-1} = e^{1.61} = 5.002\ldots \approx 5 = \text{RHS}$.)

(The expression $\ln 5 + 1$ needs to be evaluated with care. You first find the natural logarithm of 5, then add 1. The expression does not mean the same as $\ln(5 + 1)$.)

Activity 32

The number of Elvis impersonators increases by the scale factor 1.23 each year, so its doubling time in years is the solution i of the equation

$$1.23^i = 2.$$

Taking logs gives

$$\log_{10}(1.23^i) = \log_{10} 2$$
$$i \log_{10} 1.23 = \log_{10} 2$$
$$i = \frac{\log_{10} 2}{\log_{10} 1.23}$$
$$i = 3.3483\ldots = 3.3 \text{ (to 2 s.f.)}.$$

So the doubling time for the number of Elvis impersonators is about 3.3 years.

Activity 33

The number of elephants decreases by the scale factor 0.96 each year.

So its halving time in years is the solution i of the equation

$$0.96^i = 0.5.$$

Taking logs gives

$$\log_{10}(0.96^i) = \log_{10} 0.5$$
$$i \log_{10} 0.96 = \log_{10} 0.5$$
$$i = \frac{\log_{10} 0.5}{\log_{10} 0.96}$$
$$i = 16.979\ldots = 17 \text{ (to the nearest whole number)}.$$

So the halving time of the population is approximately 17 years.

Activity 34

(a) The initial level of radioactivity was 40 becquerels, and one day later it had fallen to 38 becquerels. Hence the scale factor is

$$\frac{38}{40} = 0.95.$$

(b) The half-life of the isotope, in days, is the solution i of the equation

$$0.95^i = \tfrac{1}{2}.$$

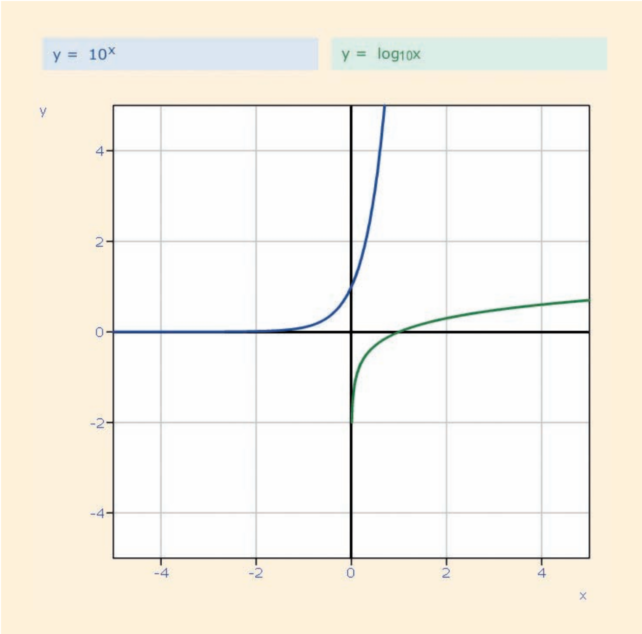
This equation can be solved by taking logs, as follows.

$$\log_{10}\left(0.95^i\right) = \log_{10} 0.5$$
$$i \log_{10} 0.95 = \log_{10} 0.5$$
$$i = \frac{\log_{10} 0.5}{\log_{10} 0.95} = 13.513\ldots = 13.5 \text{ (to 1 d.p.)}$$

So the half-life of the isotope is about 13.5 days.

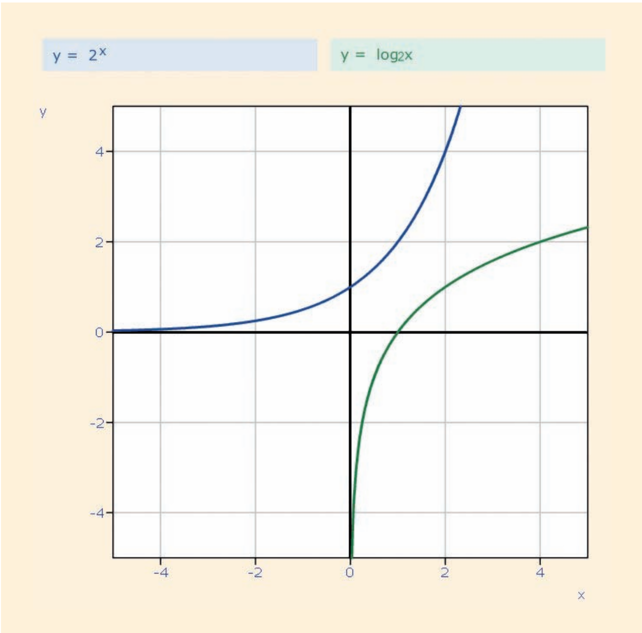
Activity 35

(b)



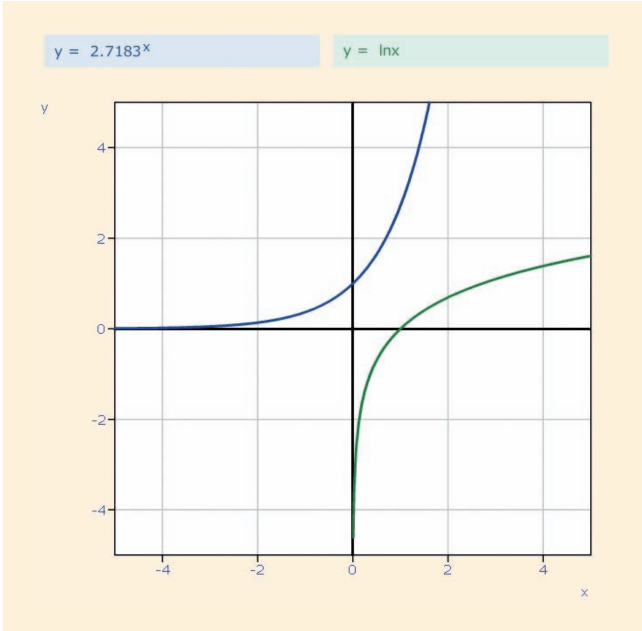
The two graphs appear to be mirror images of each other, reflected in the line $y = x$.

(c)



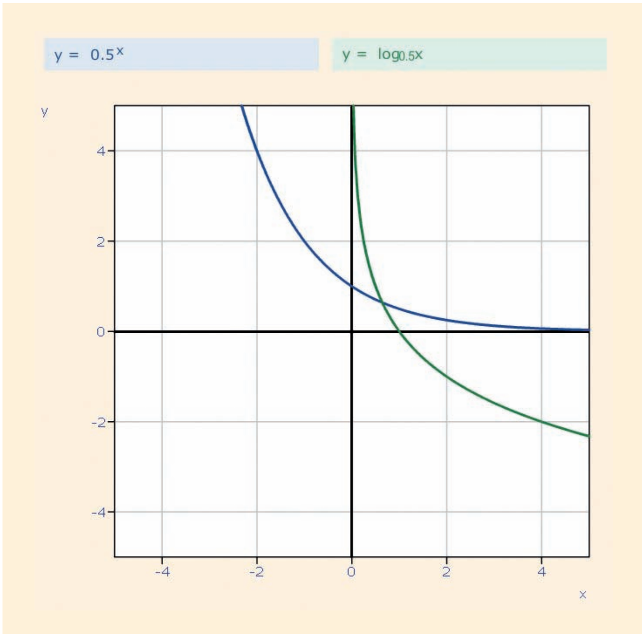
The two graphs appear to be related in the same way as in part (b).

(d)



Again, the two graphs appear to be related in the same way as in part (b).

(e)



Again, the two graphs appear to be related in the same way as in part (b).